Perspectives on Deepening
Teachers' Mathematics Content Knowledge:
The Case of the ArizonaTeachers Institute

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Number and Number Sense, A Graduate
Mathematics Course for Middle School Teachers

Abstract
The Arizona Teaching Institute (ATI) is a partnership between the University of Arizona and Tucson Unified School District. ATI has developed and implemented a Master's degree in Middle School Mathematics Leadership for middle school teachers, a Certificate in Mathematics Teacher Mentoring for secondary-certified teachers, and a postdoctoral fellowship in Teacher Preparation for recent mathematics Ph.D. graduates. The Master's degree in Middle School Mathematics Leadership is designed as a three-year, part-time degree consisting of two years of mathematics and education coursework, followed by a year of fieldwork. Courses are offered during the summer and part-time during the academic year. For example, the Numbers and Number Sense course described here addressed algorithms for operations; properties of arithmetic operations; and meanings for integers, rational numbers, and real numbers with a special emphasis on fractions. Instructors deliberately incorporated review of elementary and middle-school level mathematics concepts into explorations of more advanced topics. The course attended to pedagogical content knowledge through discussions about common misconceptions and strategies for presenting difficult material, interwoven with discussions of the mathematics disciplinary content. Although anticipated and encouraged by facilitators, these discussions were often participant-initiated, rather than planned. The sequence of mathematics courses addresses logic and proof as mathematical ways of knowing, and those learning goals are primarily dealt with in the Geometry and Algebra courses. In addition, the Numbers and Number Sense course modeled the process of making distinctions among related mathematical constructs as a way of knowing and understanding mathematics.

Introduction

The University of Arizona (UA) and Tucson Unified School District (TUSD) formed a partnership to develop and implement the Arizona Teacher Institute (ATI), a vertically integrated, nationally replicable program with the following components:

- A Master's Degree in Middle School Mathematics Leadership for middle school teachers, targeted at the large proportion of teachers who are elementary certified,
- A Certificate in Mathematics Teacher Mentoring for secondary-certified teachers,
- A Postdoctoral Fellowship in Teacher Preparation for recent mathematics Ph.D.'s.

The targets for these ATI programs are a steady state of: 11 Master's candidates beginning a three year program each summer, 2 Certificate candidates each year in this one year program, and 2 three-year post doctoral fellows. The goals of the project are to:

- Increase the number of middle school teacher leaders in the Tucson Unified School District and the Tucson area with a profound understanding of middle school mathematics and with the leadership skills to conduct effective professional development at their schools;

- Increase the number of faculty at the University of Arizona with a knowledge and understanding of how to support effective teacher preparation and in-service professional development;
- Develop and establish a permanent, replicable Master's Degree in Middle School Mathematics Leadership for producing middle school mathematics teacher leaders;
- Through the Certificate in Mathematics Teacher Mentoring and Postdoctoral Fellowship in Teacher Preparation, train and certify a corps of mathematics teacher leaders and Ph.D. mathematicians from around the country who have the knowledge and understanding to implement courses for a Master's program in their local areas; and
- Develop a distance-learning version of the Master's program that can be implemented nationally.

The Institute provides instruction in both mathematics and pedagogy in varying proportions for the different components. The focus of this article is the development of one course for the master's degree. All four core mathematics courses in the master's program are team taught by mathematics faculty and master high school teachers. The mathematics faculty will sometimes be trainees from the Postdoctoral Fellowship in Teacher Preparation, and the master high school teachers will usually be trainees from the Certificate in Mathematics Teacher Mentoring. The courses were developed in a team effort among mathematics faculty, postdoctoral fellows, and high-school teachers in the ATI program.

Master's Degree in Middle School Mathematics Leadership.

The largest component of the ATI effort is the development of a new Master's Degree in Middle School Mathematics Leadership. This master's program is designed as a three-year, part-time degree, with the target audience being middle school teachers who have elementary certification. The program, however, should be flexible enough to be a valuable educational experience for any teacher who has responsibility for middle school mathematics instruction. Further, so that the ATI program can become an integrated part of the Mathematics Department's efforts, all the new courses should be of value to any appropriate student at the university interested in the teaching of school mathematics.

The Master's degree itself consists of two years of courses in both mathematics and education taken in the summer or part-time during the year, and one year of fieldwork. The course work for the degree will consist of four four-unit mathematics courses (The Number Line, Algebra, Geometry, Probability and Statistics), two three-unit education courses (Research on the Learning of Mathematics - Mathematics Department, and Disciplined Inquiry in Education - Department of Teacher and Teacher Education), three to four units in mentoring, and six units devoted either to a practicum or a thesis, for a total of 31–32 units. In the current implementation, there is a one unit course in mentoring, and an additional three unit education course (Language and Culture in the Teaching of Mathematics.)

ATI operates under the well-supported premise that improved student achievement can be a goal and a result of professional development. In a study by Hill, Rowan, & Ball (2005), the researchers found that teachers' mathematical knowledge for teaching positively predicted student gains in mathematics achievement during the first and third grades. The authors

indicated that efforts to improve teachers' mathematical knowledge through content-focused professional development and preservice programs will improve student achievement. In a report on professional development in mathematics Mundry & Boethel (2005) pointed out that experienced teachers who know both their content and effective instructional strategies tend to produce higher academic achievement outcomes by their students. Thus, the outcomes hoped for from this project are increased student achievement as the result of increased teacher knowledge.

In more detail, the expected outcomes of this project fall into two categories: outcomes for the members of the partnership, and local and national outcomes. For the partnership, we would hope to find a graduate of the Master's Degree in Middle School Mathematics Leadership in as many of the 19 middle schools in Tucson Unified School District as possible. We expect these graduates to have the skills to help with or direct professional development with teachers at their school. As a consequence after the NSF funded part of the project, we expect increased achievement for 500 middle school students in TUSD as a result of challenging courses and curricula taught by Middle School Mathematics Leaders and the teachers they work with. Also, the Master's Degree in Middle School Mathematics Leadership will increase access by all students to highly skilled teachers and thus more challenging courses and better informed curricula. Further, as a result of the ATI interactions between candidates for Certificate in Mathematics Teacher Mentoring and the Master's Degree in Middle School Mathematics Leadership, we expect to see a lasting and growing community of mathematics educators that crosses grade levels, and school and district boundaries. The overall goal of the Arizona Teacher Institute, however, is to increase middle school teachers' access to high quality professional development through increasing mathematics faculty capacity at the University of Arizona for their involvement in teacher recruitment, teacher retention, and other teacher preparation programs.

Middle School Mathematics

Each of the four 4-unit core mathematics courses in the Master's degree (Number and Number Sense, Algebra, Geometry, and Probability and Statistics) has an emphasis on how logic and proof undergird the mathematics and provide meaning and life to it, and each incorporates technology and applications to other fields as appropriate. Roughly the courses are planned as two units of mathematics instruction, and two units devoted to how to teach the mathematics. ATI has a content advisory board consisting of two mathematicians, a mathematics education researcher, and a high school teacher. This board reviewed the initial plans for each of the content courses, and will review full content once the materials for all four courses are complete. We attempted initially to find appropriate existing course materials for each course. We considered adapting texts designed for elementary school teacher preparation such as the texts of Parker and Baldridge, Beckmann, or the materials used at the Vermont Mathematics Institute. We agreed that our final materials must satisfy the dual requirements of (a) clear, correct, and rigorous mathematics (b) support for pedagogical projects linked to state and national standards for teaching and curriculum.

Our starting point was the Arizona Mathematics Standard Articulated by Grade Level 2003 and its planned (and now implemented) 2008 Mathematics Standard Articulated by Grade Level (http://www.ade.state.az.us/standards/math/Articulated08/default.asp). These standards were the

basis of our choice of course topics. We also reflected the *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics in planning for all four of the content courses. Our primary target was middle school mathematics teachers who have elementary certification. Typically middle school consists of grades 7 and 8; however in practice, a well prepared middle school mathematics teacher needs a strong foundation in a wider spread of topics. Thus in setting the syllabi for these courses, we included topics from the standards for grades 6 and 9 as well.

Of course, the experience of the principal investigators and their preparation for the project made many curriculum choices obvious. We were often advised for the need of a thorough review of arithmetic procedures and the importance of Geometry and geometric intuition. Also, we were warned that many elementary certified teachers have had little or no training in all in Algebra, perhaps including high school. Teachers themselves often asked for help with the topics listed under probability and statistics. The refrain we heard the most was "Fractions, fractions, and then more fractions."

The Number Line

The course we designed, taught first, and have offered the most often is now called Number and Number Sense. This is the first topic detailed in the Arizona Mathematics Standards, and the content that needed to be covered was rather clear. This article will focus on this first course and reflect our experiences with the first two groups of participants.

As of January 2010, a total of 36 people have taken the course; 6 taught elementary school; 2 are middle school supervisors; 2 were candidates in a degree program other than the project's. Thus there were 26 middle school mathematics teachers in the three offerings of the course. There was no noticeable distinction between the various groups of participants except that the outside graduate students had more recent experience with some of the less familiar mathematical topics and far less actual classroom teaching experience. It should be noted that three participants have dropped out of the degree program, and one of these was the teacher at the lowest grade level in the group.

Mirroring the project goals, all content courses were set with the following objectives in mind.

Project Objectives

1. Allow middle school teacher leaders the opportunity to obtain a profound understanding of middle school mathematics and the leadership skills necessary to their professional development.
2. Increase the number of faculty at the University of Arizona with a knowledge and understanding of how to support effective teacher preparation and in-service professional development;
3. Provide additional training in the scientific and pedagogical issues of middle school mathematics for participants in the Certificate in Mathematics Teacher Mentoring and Postdoctoral Fellowship in Teacher Preparation;

4. Eventually develop a distance-learning version of all the courses in the Master's program.

Additionally we set course goals for the particular course, *Number and Number Sense*.

Course Goals

1. To build participant mastery and confidence in the use of fundamental mathematical principles and concepts involving natural numbers, integers, rational numbers and real numbers;
2. To train participants to recognize abstract mathematical constructions in middle school mathematics materials and to develop effective pedagogical methods for presenting these difficult concepts in the best way possible, and
3. To give participants the ability to read and understand new and different mathematical materials at levels both above and below where they teach, and to adapt new approaches and ideas to the middle school curriculum.

The Course

ATI faculty (A mathematics faculty member and an experienced school teacher) began by reviewing state and national middle school mathematics standards; setting out the course goals, and looking for available math texts. In a number of discussions with various experts, we were advised that many middle school mathematics teachers lacked formal mathematics training and that we should include a thorough review of the mathematics taught in elementary school as part of the mathematics content in the degree program, and this course in particular. At the same time, we wanted to be careful about presenting such a review in a way that might seem condescending to an experienced teacher. One goal of the project is to create a collegial community of mathematics educators based on mutual respect for the professionalism of all its members and this is a very important concern in designing the first course new teachers will experience in the program.

Two text books were chosen with these concerns in mind: *Mathematics for Elementary School Teachers* by Sybilla Beckmann and *Exploring the Real Numbers* by Frederick Stevenson. The Beckmann text is aimed primarily toward elementary teachers, but it will provide an excellent reference for the topics in school mathematics during and after the course. The Stevenson text takes a more mathematically sophisticated approach and is aimed at a more experienced reader. The course takes a position somewhere between the different levels of these texts. Whenever possible, the course material refers to specific sections of both texts where the topic under discussion could be found. The choice of these two texts as supplementary to the course material was made specifically with Course Goal 3 in mind.

Algorithms
We begin the course with a topic that would be new to all the participants; however, we introduce a topic more directly relevant to their teaching very quickly afterwards. Prior to the first course meeting the participants are sent a set of preliminary questions that would lead to the Euclidean Algorithm and eventually to continued fractions of rational and real numbers. The questions are presented as an exploration of repeated application of the Division Algorithm of

Natural Numbers. After a brief account of the division algorithm, the preliminary problem set presents the following task:

For a mathematical exploration, take various pairs of natural numbers a and b with a < b. The division algorithm gives a quotient and a remainder that is less than the divisor. Do not just perform the division algorithm once, but do the algorithm again dividing the just found remainder into the recent divisor. This will be possible because the remainder, when it exists, will be smaller than the previous divisor. Keep this up until the process ends, or you get stuck in some sort of rut, or it becomes clear that it will never stop on its own.

For example, start with 16 < 67. Then

$$67 = 16 * 4 + 3$$
$$16 = 3 * 5 + 1$$
$$3 = 1 * 3$$

Since there is no remainder in the last line, we have nothing to divide by. The process stops. For another example, start with 51 < 171.

$$171 = 51 * 3 + 18$$
$$51 = 18 * 2 + 15$$
$$18 = 15 * 1 + 3$$
$$15 = 3 * 50$$

Again it stops when we reach a division with no remainder.

1. Is it possible for a starting pair of numbers to never reach a conclusion in this process? Can it happen that it just produces more and more remainders and quotients forever? Can it happen that is gets stuck just repeating the same sequence of calculations over and over?
2. Is there any mathematical value in this exploration besides as an arithmetic exercise?
3. Can you think of any way of, or reason for, using this in your class?

For the most part, the ATI participants are not experienced enough with unstructured explorations to make much progress on any of these questions. Some participants do not even come up with the idea of choosing arbitrary pairs of integers to try and simply concentrate on the two examples provided. Others do make progress and even notice that the process always did come to a conclusion.

In its first meeting, the course begins with the participants discussing their ideas about this problem. Building on these ideas the instructors begin a more directed exploration of these questions. However, the purpose was not to fully investigate this process, but rather to give the participants more confidence in the power of examples in understanding an arithmetic process. The main objective, in this first class, is for the participants to completely understand that, because the remainder decreases in each step, the process must end with a division that produces no remainder. We immediately stress the value of abstract reasoning in understanding mathematics as we work toward Course Goal 2.

Before all the questions raised by this first process are completely resolved, the next topic is introduced in a way analogous to the first.

The repeated division process we just investigated started with $a < b$, and after that looks like this:

Step 1: $b = aq_1 + r_1; r_1 < a$
Step 2: $a = r_1q_2 + r_2; r_2 < r_1$
Step 3: $r_1 = r_2q_3 + r_3; r_3 < r_2$
Step 4: $r_2 = r_3q_4 + r_4; r_4 < r_3$
Step 5: $r_3 = r_4q_5 + r_5; r_5 < r_4$
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It is called the Euclidean Algorithm. It is not part of the typical middle school curriculum, but it is an incredibly important idea that has real everyday applications.
There is, however, another way to repeat the division algorithm that is definitely part of the curriculum that you teach every year. This process also starts with any a and b, and repeats as

Step 1: $b = aq_0 + r_1; r_1 < a$
Step 2: $10r_1 = aq_1 + r_2; r_2 < a$
Step 3: $10r_2 = aq_2 + r_3; r_3 < a$
Step 4: $10r_3 = aq_3 + r_4; r_4 < a$
Step 5: $10r_4 = aq_4 + r_5; r_5 < a$
:::::   :::::::

The participants are asked the same three questions about this second process. This time, however, the participants seem quicker to note that there is no guarantee that the process will stop. As it turns out, this process amounts to the decimal expansion of a rational fraction, and in time, the participants recognize it as such. Once this occurs, the exploration of this second process leads naturally into a careful mathematical analysis of the typical long division algorithm taught in school. The desired review of this process then appears as a natural part of an exploration of a new idea. We try to follow this template while introducing every review of school mathematics, always presenting the review in the context of a new problem or a different approach.

Rigor
The major mathematical issue we grappled with while designing the course was how much we should justify the mathematics we covered and what level of rigor we should use. The Arizona State Standards include a strand devoted to Structure and Logic. These standards include specific requirements in 7th and 8th grade like:

- Analyze a problem situation to determine the question(s) to be answered.
- Communicate the answer(s) to the question(s) in a problem using appropriate representations, including symbols and informal and formal mathematical language.
- Solve logic problems involving multiple variables, conditional statements, conjectures, and negation using words, charts, and pictures.
- Demonstrate and explain that the process of solving equations is a deductive proof.
- Identify simple valid arguments using if… then statements.

The high school standards for grades 9 and 10 include:

- Draw a simple valid conclusion from a given if…then statement and a minor premise.
- Construct a simple formal deductive proof.
- Identify and explain the roles played by definitions, postulates, propositions and theorems in the logical structure of mathematics.

Again we choose a gradual approach and introduce rigor in a practical context that illustrates how it clarifies observations made in specific calculations. For example, the greatest common divisor of two natural numbers is an important notion in middle school mathematics. Its name gives its definition, and students are taught to compute it using factorization. To find the common divisors of 28 and 42, one finds the prime factors of both. To start, a student writes 28 as a product of divisors, say 4 and 7. Now the divisor 4 can be written as the product of 2 and 2, but 7 has no proper factors. In many middle school curricula, the result is displayed as a "factor tree," but mathematically it just means 28= 2*2*7 where all three divisors are primes. In a similar way, 42 = 2*3*7. Middle school students are taught to assemble the greatest common divisor of these numbers by choosing all the prime factors the numbers have in common: 2*7=14. Indeed 14 is a common factor of 28 and 42, and no other common factor is greater than 14. Notice, however, that the process of computing the greatest common divisor does not actually illustrate the fact that 14 is the largest number in the list of numbers which divide both 28 and 42.

The Euclidean Algorithm gives another (more efficient) method for computing the greatest common divisor. In the continued directed exploration of this process, participants are led to uncover this fact. This method of finding the greatest common divisor is more efficient than factorization, but it rarely covered in schools. The ATI participants see this as an amazing result.

We ask the participants to discover the connection between the Euclidean Algorithm and greatest common divisors by tracing all the common divisors through a specific short example that also illustrates the name "greatest common divisor":

Starting with the numbers 672 and 420, we compute

$$672 = 420 * 1 + 252,$$
$$420 = 252 * 1 + 168,$$
$$252 = 168 * 1 + 84,$$
$$168 = 84 *2 + 0.$$

Now compute:
1. the common divisors of 672 and 420
2. the common divisors of 420 and 252
3. the common divisors of 252 and 168
4. the common divisors of 168 and 84
5. the common divisors of 84 and 0.

To get started with the first problem, participants must review the process of factoring integers into primes and methods for reassembling those primes into divisors. The prime factorization of 672 is 2*2*2*2*2*3*7; and its complete set of divisors is

{1, 2, 3, 4, 6, 7, 8, 12, 14, 16, 21, 24, 28, 32, 42, 48, 56, 84, 96, 112, 168, 224, 336, 672}.

The prime factorization of 420 is 2*2*3*5*7; and its complete set of divisors is

{1, 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 105, 140, 210, 420}.

The divisors common in these sets are

{1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84}.

In fact, this collection of common divisors is the answer to all five questions. To see that this is the case in question 5 the participants need to recognize that all natural numbers are factors of 0. That tends to lead to a class discussion about the value in making what at first seems to be a rather innocuous observation.

We expect that the obvious pattern will emerge quickly enough that participants will avoid a lengthy calculation. However, when the participants use this observation, the instructors immediately ask exactly why it works. Participants are led to the observation that, when $b = aq + r$, the common divisors of b and a are also the common divisors of a and r. This can be applied repeatedly to see that the common divisors of the pair 672 and 420 are eventually the common divisors of 84 and 0. The main goal of this exercise is not to help participants appreciate the labor to be saved by seeing this pattern. This exercise is meant to help the participants reach Course Goal 2, i.e., to train participants to recognize abstract mathematical constructions in middle school mathematics materials.

The important mathematical idea behind this is the distributive property of arithmetic. For all numbers $a*(b + c) = a*b + a*c$. The Arizona Mathematics Standards include the all fundamental arithmetic properties in every grade past 6: the commutative property, the associative property and the distributive property. The commutative and associative properties of addition and multiplication are important, but when they occur in arithmetic, their implications can seem obvious. The distributive property often provides additional meaning to an arithmetical expression, and it can lead to informative conclusions. The distributive property is behind all sorts of middle school mathematics, from the normal algorithm for long multiplication to the well known "FOIL" method in algebra. This property is used so frequently, however, that it is easy to miss if you are not looking out for it. In the exercise above, ATI participants are directed to discover this powerful practical use for this simple abstract idea. Our purpose is to train the participants to recognize the distributive rule when it occurs in any context.

This careful analysis of the Euclidean Algorithm is a major step in building the profound knowledge of middle school mathematics that is a project goal. This is the first really concrete example where a fundamental mathematical concept, the distributive rule, has a direct connection to a familiar middle school problem. The application, however, is not at all familiar to the participants. Thus covering the Euclidean Algorithm fits naturally into Course Goal 1.

Finally to illustrate this practical value of the Euclidean algorithm, we assigned several arithmetic problems where it can be applied:

Reduce the following rational numbers into lowest terms:

1. $\frac{99352576}{55754176}$

2. $\frac{99352576}{22379571}$

3. $\frac{118627}{120143}$

4. $\frac{116749}{120143}$

Solution 3:

Using the repeated division process with $a = 120143$ and $b = 118627$, compute:

Step 1: Find q_1 and r_1 that satisfy $b = aq_1 + r_1$
$118627 = 120143*0 + 118627$
so $q_1 = 0$ and $r_1 = 118627$

Step 2: Find q_2 and r_2 that satisfy $a = r_1q_2 + r_2$
$120143 = 118627*1 + 1516$
so $q_2 = 1$ and $r_2 = 1516$

Step 3: Find q_3 and r_3 that satisfy $r_1 = r_2q_3 + r_3$
$118627 = 1516*78 + 379$
so $q_3 = 78$ and $r_3 = 379$

Step 4: Find q_4 and r_4 that satisfy $r_2 = r_3q_4 + r_4$
$1516 = 379*4$
so $q_4 = 4$ and $r_4 = 0$

The last non-zero remainder in the series, 379, is the greatest common divisor of 120143 and 118627. Dividing both the numerator and denominator by 379, we find that
$\frac{118627}{120143} = \frac{313}{317}$

The students are later shown how to use continued fractions to compute the reduced fraction from the partial quotients. The method is based on the expression of a real number as a continued fraction, which for rational numbers can be found using the Euclidean algorithm and written as:

$$\frac{b}{a} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{...}}}$$

Calculating only a portion of this continued fraction provides an approximation of the actual number, and students are shown a shortcut using a table:

		q_1	q_2	q_3	
A=0	B=1	A+B*q_1=C	B+C*q_2=D	C+D*q_3=E	…
R=1	S=0	R+S*q_1=T	S+T*q_2=U	T+U*q_3=V	…

The results C/T, D/U, E/V, … give increasingly accurate approximations of *b/a*. For rational numbers the sequence ends and the reduced fraction appears in the final column of the complete chart. Continuing with the example shown in the table, we use the partial quotients 0, 1, 78, and 4 to get:

$$\frac{118627}{120143} = 0 + \cfrac{1}{1 + \cfrac{1}{78 + \frac{1}{4}}}$$

The reduced fraction 313/317 appears in the final column of the chart:

		q_1=0	q_2=1	q_3=78	q_4=4
		A+B*q_1=C	B+C*q_2=D	C+D*q_3=E	D+E*q_4=F
		0+1*0=0	1+0*1=1	0+1*78=78	1+78*4=313
A=0	B=1	C=0	D=1	E=78	F=313
		R+S*q_1=T	S+T*q_2=U	T+U*q_3=V	U+V*q_4=W
		1+0*0=1	0+1*1=1	1*78+1=79	1+79*4=317
R=1	S=0	T=1	U=1	V=79	W=317

Real numbers
Probably the greatest pedagogical problem we faced in designing the course was choosing an approach to the full set of real numbers that was mathematically accurate, applicable to the middle school curriculum, and practical enough to add understanding yet avoid adding confusion. The Beckmann text takes a very concrete and computational approach to this that is completely appropriate for elementary teachers, but that does not provide sufficient rigor to address the technical issues that can come up in later grades. The Stevenson text gives two statements of the completeness axiom. The first, "A decimal expansion represents an existing number," is entirely intuitive and not far beyond Beckmann's approach. Stevenson also includes the very formal definition: "Every rational sequence of Cauchy sequences has a limit." This is followed with a full mathematical definition of a Cauchy sequence.

It took mathematics several thousand years to come to complete grips with the intricacies of the real numbers. The decimal number system has the great advantage of hiding many of these intricacies. However, these technical issues remain, and they can appear in the middle school classroom. Early in the course we ask the participants a few questions that touch these issues:

> 1. Of the numbers 0.5 and 0.49999999… (where the 9's go on forever), which is the larger?
> 2. What is π?
> 3. Compute $\sqrt{2}$ to 4 decimal places?
> 4. What is 1.4142^2?

The first question is meant to spark a common debate, and many participants will begin to wonder if 0.49999… is a real number at all. The second question is meant to continue that debate. The typical answer that people agree on is: "It is 3.1415926… where the decimal goes on forever without a pattern." The ensuing discussion about what that actually means exposes a great deal of confusion about infinite decimals. The answer to the third question is 1.4142, but the answer to the fourth is not 2. It rounds off to two, but only if you do so properly.

A purely computational definition of infinite expansions, like that in Beckmann's book, will explain the answer to the first question, and with work, will help with the last pair. However, π is still a problem. A purely intuitional approach causes trouble in question 1, but it makes π a bit easier to swallow. In a practical sense, the goal for middle school students is to develop a reasonable intuition about infinite expansions and exact real numbers strong enough to provide them with fluid computational abilities with their approximations. However, if we are to provide middle school teachers with a more profound understanding of the real numbers as a course goal, as project objective 1 requires, then we must look beyond the middle school curriculum.

The best solution seems to be the introduction of a mathematically sound definition of the Real Numbers.

There are a number of mathematical definitions of the real numbers, but the text we use, Stevenson, uses Cauchy sequences. This may be a bit too rigorous for the needs of our participants; so we choose a pared down version. The explorations of continued fractions and decimal expansions of rational numbers lead naturally to the ideas of upper and lower bounds of numbers. We take advantage of this and cover these terms carefully as they appear. Then when we are ready to deal with real numbers, we refer to Stevenson's definition of completeness. We define a (pseudo) Cauchy sequence to be an alternating sequence of upper and lower bounds with a difference that approaches zero. Since continued fractions of numbers like $\sqrt{2}$ produce such sequences, this abstraction can be presented in the context of an increasingly familiar topic. The partial quotients for $\sqrt{2}$ obtained from the real number version of the Euclidean algorithm are 1, 2, 2, 2, 2, …, where the 2's go on forever. The fraction chart one obtains from the partial quotients is:

		1	2	2	2	2	2	2	….
0	1	1	3	7	17	41	99	239	….
1	0	1	2	5	12	29	70	169	….

This method yields a sequence of increasingly accurate fractional approximations of $\sqrt{2}$ that are alternately larger and smaller than the actual value.

Thus Stevenson's version of the Completeness Axiom, "Every rational sequence of Cauchy sequences has a limit," provides a rigorous definition of real numbers in the context of the course.

This definition has the advantage of being mathematically accurate. (It is basically the Closed Interval Theorem.) At the same time, it also has practical applications. Knowing a real number under this definition means knowing an infinite sequence of upper and lower bounds. An arithmetic problem involving real numbers can be solved by using these bounds to find a Cauchy sequence for the answer. Although this is a theoretical result, it provides a practical method for approximating solutions.

For example, participants are asked to consider the real number given by the infinite decimal: 0.12121212… This expansion indicates the infinite sequence of upper and lower bounds:

Lower	Upper
0.1	0.2
0.12	0.13
0.121	0.122
0.1212	0.1213
0.12121	0.12122

According to the completeness axiom, this (pseudo) Cauchy sequence identifies a single real number. Whatever this number is, it can be multiplied by 33. Since all the lower bounds are lower than the number, multiplying each them by 33 will give a lower bound on the product. The upper bounds work the same way. Thus $33 \times (0.12121212\ldots)$ is given by the sequence

Lower	Upper
3.3	6.6
3.96	4.29
3.993	4.026
3.9996	4.0029
3.99993	4.00026

This Cauchy sequence produces the number 4. Thus, by the definition of the real numbers, $33 \times (0.12121212\ldots) = 4$. We must have $0.12121212\ldots = \frac{4}{33}$. Participants are then asked to verify this equality in several more familiar ways, including the methods given in Beckmann's text.

Thus an application of this definition leads naturally to a thorough discussion of decimal round-off and approximation. This in turn makes it easy to motivate a detailed look at the impact of approximation on the accuracy, and the meaning of calculator and computer calculations.

Technology
The final decision in the design of the course was how technology should be integrated into the material. We do stress the use of calculators, but we simply use them as a calculation tool when needed in the course. The plan is that graphic calculators play an important role in later the Algebra and the Probability and Statistics courses. Instead of using calculators to illustrate calculations in the Number class, we use a spreadsheet and computer projection. Participants enter the ATI program with a wide spread of experience with mathematical technology. Even the most experienced, however, do not understand the illustrative power of a spreadsheet in lower level arithmetic problems. From the beginning of the course, examples are illustrated using a project work-page from a spreadsheet. The participants immediately see the advantages of this kind of presentation to produce a number of examples rapidly. And although it not listed in the course syllabus, spreadsheet design becomes part of the course. Further, there were many classroom discussions about the effective use of spreadsheet in the middle school classroom and how this technology can be used to improve pedagogy. The introduction of this increasingly common form of computer programming into the course fits directly into course goals 2 and 3.

This extra time devoted to technology is seamlessly integrated into the course and is used to help reinforce the participants' command of arithmetic. In a homework exercise, participants are asked to compute 2^{100} as accurately as possible by any means possible. The weakest participants often take this as a very simple question to answer, and they give answers like 1.267651E30 straight off a calculator. For the most part, the participants recognize that this answer is in scientific notation and know that it means 1267651000000000000000000000000. Of course, not all answers in this form are going to be in complete agreement, and this prompts a discussion of what "as accurate as possible" means. It seems inevitable that in a class of middle school teachers that, without prompting, at least one person will note that an accurate value of 2^{100} cannot end in a 0.

If the instructors can restrain themselves at this point, the rest of the class will ask this person to explain their reasoning. Further, participants soon discuss the idea that even an integer might be approximated. The ensuing participant driven conversation works directly toward the first course goal: to build confidence in the use of fundamental mathematical principles to explain the meaning of a calculation result. The original question draws participants to recognize the implicit approximation in scientific notation and the implications that approximate calculation has for exact results and the problems caused by large numbers.

The participants themselves express a need for the next logical step, finding an exact value of 2^{100}. The course materials then lead the instructors to design a spreadsheet that does multi-digit addition and multiplication. This is done on a spreadsheet in base 1,000,000. To set up this spreadsheet, the class first must carefully analyze the grade school algorithms for addition and multiplication in base 10. After a careful and complete review of the familiar decimal hand algorithms for these operations and why they work in base 10, the construction of the spreadsheet begins. We do devote classroom time to constructing a working spreadsheet; however, we eventually provide a version to replace the one designed in the classroom. (This allows us to ignore certain formatting issues during the class and concentrate on the mathematics of the spreadsheet rather than its appearance.) The participants are now ready to compute 2^{100} exactly.

Fractions

There is no question that a complete and thorough account of fractions and rational numbers needs to be a major part of this course. A major pedagogical problem in teaching fractions is that fractions have several, closely related, interpretations. In some sense, the objective in teaching fractions to school children is to bring them to a point where they can move effortlessly between these interpretations. However, the differences between these interpretations are significant and can easily lead to confusion in learners.

First, fractions give the number of equal parts of a whole. However, even under this very first introductory meaning of the term, a fraction productively be considered as a division problem. While $\frac{21}{7}$ still means 21 pieces where 7 pieces are a whole that amounts to 21 divided by 7 which is 3 wholes. By middle school, fractions are also used to represent quantities; that is, rational numbers. This means that $\frac{21}{7} = 3$ because:

1. If wholes are divided into 7 equal pieces, than three wholes produce 21 pieces.
2. The fraction $\frac{21}{7}$ is equivalent to the fraction $\frac{3}{1}$.
3. The expressions $\frac{21}{7}$ and 3 are the same rational number. Thus a fractional expression can be interpreted as either a division problem, an expression for parts of a whole, or a rational number.

The Arizona Math Standards set a timeline for students to understand these three interpretations. In Grade 3 they call for students to "Express benchmark fractions as fair sharing, parts of a whole, or parts of a set."

However, any third grader intuitively understands the important distinction between $\frac{1}{2}$ of a whole cookie and $\frac{1000}{2000}$ of a whole cookie. In terms of "fair sharing", these may be equivalent, but they are not the same. The standards stress the understanding of fractions as part of a whole, and how this is related to the operation of division. Then in Grade 7, these standards call for students to "Compare and order rational numbers using various models and representations."

Thus a successful 7th grader and should no longer just see the fractions $\frac{1}{2}$ and $\frac{1000}{2000}$ as representing equivalent quantities, but they should be able to identify them as equal rational numbers. The student should still realize that these are not exactly the same when viewed as part of a whole, but they represent equivalent quantities. But in 7th grade, because they are equivalent as fractions, students are expected to call these equal rational numbers. The distinctions are subtle, and potentially confusing. Certainly to meet these goals, teachers must teach their students to move fluidly between these three possible interpretations of fractional notation. However, to get students to this point, the teachers need to be quite aware of these different points of view and be ready and able to offer explanations using any one. The teachers' own ability to interpret fractions in slightly different ways is needed to support their ability to help their students.

To increase participants' awareness of subtlety of statements about fractions, ATI participants are asked to fill out an online survey. The survey consists of 25 statements involving fractional notation. Participants are asked to judge whether they believed the statement was (a) mostly a

statement about division, (b) mostly a statement about fractions, or (c) mostly a statement about rational numbers. Some of the statements can be interpreted as any one of these, so the survey does accept multiple correct answers. At the same time, participants are told to select an answer that they consider the best interpretation. One intention of the survey is to foster in participating teachers a mathematician's way of thinking about distinctions between mathematical constructs. For example, the statement, "The denominator of $\frac{5}{13}$ is 13," is a statement about a fraction, not about a rational number, because there is no single fraction that uniquely represents the rational number that can be written as $\frac{5}{13}$. Some of the statements on the survey are:

1. $\frac{51}{17} = 3$	9. $35 \div 14 = \frac{5}{2}$
2. $\frac{55}{22} = \frac{5}{2}$	10. $\frac{1125}{2249} \cong \frac{1}{2}$
3. $\frac{55}{22} \equiv \frac{5}{2}$	11. The denominator of $\frac{5}{13}$ is 13.
4. $\frac{27}{11} = 2\frac{5}{11}$	12. $\frac{51}{13}$ is improper.
5. $\frac{11}{8} = 1.375$	13. $\left(\frac{5}{12}\right)^2 = \frac{25}{144}$
6. $\frac{15\pi}{3\pi} = 5$	14. $\left(\frac{5}{12}\right)^{-1} = \frac{12}{5}$
7. $\frac{1}{7} = 0.\overline{142857}$	15. $\frac{3+2\sqrt{2}}{1+\sqrt{2}} = 1 + \sqrt{2}$
8. $\frac{2}{3} \cong 0.6667$	

The survey results become part of an open class discussion. Participants are asked to defend their choices for each question. There are discussions about what answers are "right" and "wrong." There are discussions about how an expression could and should be interpreted for clarity. The discussions involve mathematical issues and pedagogical issues, and inevitably lead to the question of which issue is more important.

As a conclusion to this exercise, the participants were given a writing assignment to help them articulate their understanding of the distinctions between fractions and rational numbers and the importance of those distinctions in doing mathematics and in teaching mathematics. The prompts for the writing assignment were:

1. How important is it to distinguish between fractions and rational numbers in practice?
2. How important is it to distinguish between fractions and rational numbers in teaching?
3. What exactly is a rational number?

This exercise is aimed squarely toward meeting Goal 2 of the course: to train participants to recognize abstract mathematical constructions in middle school mathematics materials and to develop effective pedagogical methods for presenting these difficult concepts in the best way possible. The intrinsic abstract nature of fractions is easily lost to someone who gains the working knowledge of them that is expected of a high school student. Yet, it is exactly the loss

of understanding multiple interpretations that makes fractions such a challenge for students into their college years.

Evidence of Impact

The preliminary results of the project evaluation influenced the direction of the design of the number course as they became available. This section includes excerpts and summaries of their reports.

The evaluation included of an investigation of participating teachers' content knowledge using the Middle School Content Knowledge Survey, a questionnaire to determine implementation progress, self reporting by participants, surveys, ratings by principals, determination of academic achievement in the participants' classes through comparison of the standards-based Arizona Instrument to Measure Standards (AIMS) and the TerraNova standardized achievement test, and observations of the participants using the Reformed Teaching Observation Protocol (RTOP) and an Effective Instruction Rubric (EIR). Initial data were collected from all entering participants, and yearly throughout their participation in the program.

In this article, we concentrate on the three aspects of this ongoing evaluation: impact on classroom teaching techniques, leadership, and content knowledge. As one would expect, the best information is available on the participants from the first offering of the course, and we focus on the results from this group to which we refer as Cohort One.

Initially, the cohort showed improvement in Number Concepts content knowledge, as measured by the IRT, with 7 of the 10 people involved demonstrating increased knowledge between the pretest and the first year. By year two, however, the (normed) results had become more mixed. Here 6 of the 10 still showed improvement over the pretest, and the range of the results increased. Participants who had shown the most improvement immediately after the number course fell back; while participants whose improvements were less dramatic continued to improve in year 2. This cohort will complete the three year program in spring 2010. In contrast, the self reports showed that the participants themselves felt strongly that their knowledge steadily increased.

The data collected on leadership presents some difficulties. Both cohorts that completed their first full year showed an overall drop in leadership skills as measured by principal surveys, even though the leadership rubric used showed some small numerical improvement. This could well be the result of the time demands on the participants; however, comparing the results after year one, it seemed that Cohort One had more difficulty maintaining and increasing their leadership skills than did Cohort Two. Self reporting indicated that the participants had high expectations for themselves and their students because of their participation in the program. Also, many participants managed to find the time to attend and even present at meetings of educators.

The most positive results come from the measures of classroom teaching practice. Both the RTOP and EIR measures showed marked and steady increases. As described by the developers, the Reformed Teaching Observation Protocol (RTOP) was developed as an observation instrument to provide a standardized means for detecting the degree to which K-20 classroom

instruction in mathematics or science is reformed. It does not presume that reformed instruction is necessarily quality instruction. The EIR is another observational protocol that was designed by the project's evaluation team to measure the use of classroom practices that are well accepted and research has shown are effective. Using RTOP in the area of lesson design, participants increased their scores by 48%; in the area of content, they increased by 58%; and in class culture, the increased 59%. In EIR the cohort gained 35%. Several participants showed a large gain in instructional practices, while others made modest gains; however, it was those scoring high at baseline that reflected the smaller gain. Self reports echoed this. Incidentally, the impact of the Education courses Cohort One took appeared here as well with all participants reporting that they used research in their teaching either frequently or regularly.

Lessons Learned

We learned a great deal when we offered the courses for the first time. Many of the teacher participants expected a stronger workshop format, and they expected to be given materials that they could take directly back to their class. Of course, the intention was to create teacher leaders, and it took a while to get the participants to understand this. As the course progressed, participants did come to realize that every mathematical topic was accompanied by a discussion of its classroom implications. These discussions of new mathematics and new approaches to familiar mathematics served to ground the course for those participants looking for more immediate practical ideas. These discussions were led by the high school teacher, and they often led to vigorous exchanges of ideas among the participants. It was easy for these discussions to wander into other teaching and classroom problems. However, the high-school teacher became quite adept at directing the discussions at critical moments and keeping them focused on the mathematics. Even though each topic was introduced and concluded with a collection of both mathematical and pedagogical questions, the participants were much better at introducing the educational issues that came with the mathematics. Also the distinct differences in the academic approaches between college students and returning adult learners had a greater impact on the course that the first faculty member instructors expected. Future instructors will be cautioned about this issue.

From the beginning of the design of this course, we were told that fractions were a major difficulty in the middle school curriculum. Even planning for this, we discovered that addressing all the conceptual and practical problems students of all ages have with arithmetic fractions is extremely difficult. Many textbook and curriculum authors have addressed this difficulty in various inventive ways. Yet it appears that no one approach ever solves the entire problem.

In our case, we felt that we had made some progress with the participants in this Number and Number Sense course. However, the preliminary results of evaluation indicate that, to some extent, the progress is not long lasting. The lesson we take from this is that we cannot delegate the arithmetic of fractions to only one of the four mathematics content courses we require. We need to make certain that we reinforce and extend participants' knowledge of fractions in the curriculum we develop for the other three courses: Algebra, Geometry, and Probability and Statistics.

The real numbers and infinite decimal expansions are the source of a great deal of confusion. Middle school teachers, especially those whose mathematical training emphasized elementary school concepts, have at best the general intuitive understanding of these topics of the general population. Yet all the teachers we dealt with could cite examples where their students were confused by some aspect of these abstract ideas and where they felt unable to help. Class discussions revealed a great many misperceptions about infinite decimals, and it was clear that most of the participants did not have a deep enough understanding of the issues involved to even recognize the problems. It does not appear that the mathematics education literature has much guidance to offer about dealing with technical issues that can arise from infinite decimals and approximate calculation in either the middle school curriculum or in teacher training. Yet with the ubiquitous use of calculators and other machine arithmetic, these issues are unavoidable in the middle school classroom; students do ask questions like:

- Is $\frac{1}{3} = 0.33333333$ or do the threes go on forever?
- Is 0.9999… the same as 1.00000…?
- How come the answer 0.714286 I get on the calculator is not the same as the answer $\frac{5}{7}$ in the book or the 0.714285 that I got by hand?

A middle school teacher needs to have answers for these questions, but how much about those answers does the teacher need to understand? A high school certified teacher probably has enough background and formal mathematical training to handle these types of problems. A middle school teacher, however, may only have the common general knowledge that, in practice, these issues are not a serious problem. Using good pedagogy, including distinguishing between a student exploring the underlying mathematics and one who simply wonders why the calculator produced an unexpected answer, a teacher may be able to develop answers to these kinds of student questions.

But how much of the mathematics does the teacher need to understand to be able to use good pedagogy? One lesson we learned in developing and teaching this course is that this is not an easy question to answer. Mathematics is not just about convention; so 0.9999… is not equal to 1 because mathematicians say so. Mathematics is more precise than that. However, it may be that 0. 714286, $\frac{5}{7}$, and 0.714285 are all acceptable answers to a problem because they are close enough. Where is the precision in this? Eventually we hope that middle school students become comfortable enough with numbers that they can ignore these problems like most other people. Middle school teachers are tasked for helping students reach this point, but can they do it by ignoring those problems themselves?

Our hypothesis is that they cannot. We covered real numbers from a relatively abstract point of view to clearly illustrate the logical difficulties behind numbers we all take for granted. We spend a good deal of time on approximate calculation and its impact on extended calculations. We had the participants track errors caused by rounding and chronicled the explosion of mathematical error that division can cause. Our goal was to be sure that the participants understood the underlying technical issues of calculation, that they recognized both the perils of explanations that were too detailed and the confusion that too general an explanation can cause. We wanted the participants to understand exactly why a calculator is not the final authority on the result of a calculation.

The main lesson is that there is more to the teaching of basic number concepts than simple arithmetic and arithmetic operations. The conceptual aspects of understanding numbers and quantities present a major challenge to teachers of all levels.

Finally, it is clear that many dedicated teachers across the country return to school to obtain higher degrees through night and summer classes. And they do so very successfully. However, it is a large mistake to discount how incredibly difficult this actually is. First, the demands of a normal teaching load on a middle school teacher of any sort are great and extend well past their responsibilities in the classroom. Also, at the point in a teacher's career where a higher degree will have the most impact, they are often at a point in their lives where family responsibilities are at their greatest. The fact that anyone in this situation would be willing to take on even one technical mathematics course and actually put in the effort to learn the material covered is astounding.

A STEM faculty member, a mathematics educator, or a school administrator can quickly determine what the content of a professional development course for in-service teachers should be. An experienced university instructor knows how to put a course together that meets that set of academic goals. However, if that course is meant to be part of a program that takes place on top of a teacher's normal schedule, the design of a course is much more difficult. Designing an effective and valuable course for a part time student requires an understanding of the real needs of the students. In our case, the participating teachers responded to the demands of our course with an admirable and remarkable dedication. However, it was clear that these demands took a toll on them. For professional development programs that will have lasting positive impact, we cannot rely entirely on the work ethic of the teachers. State education departments, universities, colleges, districts, and individual schools need to work together to make professional development a part of teachers' regular workloads, with appropriate time, support, and compensation.

References

Beckmann, S. (2005). *Mathematics for Elementary Teachers*. Boston, MA: Addison-Wesley.

Hill, H. C., Rowan, B, and Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Mundry, S, & Boethel, M. (2005). What experience has taught us about professional development. The Eisenhower Mathematics and Science Consortia and Clearing House Network. Available from http://www.sedl.org/pubs/ms90/experience_pd.pdf

Stevenson, F. W. (2000). *Exploring the real numbers*. Upper Saddle River, NJ: Prentice Hall.