# Perspectives on Deepening
# Teachers' Mathematics Content Knowledge:
# The Case of the Greater Birmingham Mathematics Partnership

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John C. Mayer
Rachel D. Cochran
Bernadette Mullins
Ann M. Dominick
Faye B. Clark
Jason S. Fulmore
Ruth Parker
Patty Lofgren

Abstract

The Greater Birmingham Mathematics Partnership (GBMP) is a collaborative effort involving two institutions of higher education, University of Alabama at Birmingham and Birmingham-Southern College; one private non-profit organization, the Mathematics Education Collaborative (MEC); and nine Greater Birmingham, Alabama school districts.  The project's work included professional development for in-service teachers; teacher leader development; revised coursework for pre-service teachers, including a new track for mathematics majors and accompanying new certification for middle school mathematics teaching; and community outreach.  GBMP conducted seven mathematics content courses, offered as two-week summer immersion experiences to teachers in the partner districts. Additional professional development for teacher-leaders was provided during the school year. *Patterns: The Foundations for Algebraic Reasoning* was required as a prerequisite to all other courses. In it, mathematics teachers explored patterns in an inquiry-based environment using a variety of representations including graphs, tables, geometric models, algebraic expressions, and verbal contexts. Course participants were K-20 teachers of mathematics who learned mathematics together, which allowed them to see how to meet a range of learning needs and to see how the foundation of algebraic reasoning is laid throughout the grades. They also discussed articles chosen to help participants better understand the theoretical underpinnings of constructivist theory and the implications for classroom practice.   Throughout all of the courses, participants engaged in generalization and justification as mathematical ways of learning and knowing.  Over the course of the five-year project several local teacher-leaders and higher education faculty associated with the GBMP completed an internship process to become MEC instructors, and they now teach the courses. In the final year of the project, school-based professional learning communities were initiated to support teachers in implementing inquiry-based instruction and performance assessments.

## Context and Focus of GBMP Professional Development

In 1990 a group of eight teachers who had studied Piaget's theory about how children learn were meeting regularly after school to talk about how this theory impacted their practice. They helped each other learn to listen to students and adjust their questions based on what students understood. They supported each other in implementing inquiry-based teaching. These teachers soon decided that to influence and educate their colleagues and administrators about what they were doing, and to have a broader impact on students, they would host teaching conferences, and the Birmingham Constructivist Teachers' Network (the Network) was formed. These annual one-to-three-day conferences, which sometimes drew as many as 500 teachers from across the nation, were held for ten years from 1990 to 2000.  Nationally-known educators such as Deborah Ball, Virginia Bastable, Eleanor Duckworth, Cathy Fosnot, Constance Kamii, Cornelia Tierney, and others, gave the keynote addresses and Network teachers conducted the breakout sessions.

Dr. Ruth Parker, founder and CEO of the Mathematic Education Collaborative (MEC), was the keynote speaker at the ninth and tenth annual conferences. In her keynote address, she shared information about her work that involved in-depth mathematics content courses for teachers and educating all the stakeholders in a community about quality mathematics education.  After these conferences, the Network teachers decided that to continue to grow as mathematics teachers, they needed more in-depth mathematics instruction themselves. As these teachers continued to

implement inquiry-based instruction, they found that they needed more content knowledge to be able to follow students' thinking and take students as far as they could go mathematically. MEC, a non-profit located in Ferndale, Washington offers a series of nine-day mathematics content courses for teachers based on a constructivist and social constructivist theory of learning. Additionally, they have a community engagement model that provides learning opportunities over time for parents, teachers, administrators and community members as well as a model for developing cadres of local teacher-leaders. The work that the Network had done provided fertile ground for extending MEC's work into the Birmingham area. In 2001-2002, educators in the Hoover and Mountain Brook systems decided they wanted to replicate MEC's successful work in Colorado and Oregon in the Birmingham area. They presented this idea to representatives of other Birmingham area school systems, UAB, BSC, and MEC. During the subsequent year, representatives engaged in discussions about their shared needs and the potential for producing a partnership that would successfully address these concerns. The Greater Birmingham Mathematic Partnership (GBMP) was formed with the goal of improving mathematics education by enhancing teacher content knowledge, pedagogy and assessment, and educating the greater Birmingham community about quality mathematics education.

Because of strong local interest in GBMP and its goals, local grant funds were obtained in spring 2002 to support a pilot phase of this project. MEC's mathematics content course, *Patterns: Foundations for Algebraic Reasoning,* a nine-day, in-depth mathematics content course, was scheduled for K-12 teachers in the summer and filled immediately. A second course was added, and it promptly filled. MEC offered an evening session on mathematics education for the public that was well attended and enthusiastically received. Along with the school districts, local businesses and foundations funded summer courses for the next two years. Again, the summer courses were oversubscribed and received overwhelmingly positive reviews.

Based on the successful pilot efforts with K-12 teachers, GBMP applied for and was awarded a $9.96 million NSF MSP grant in September 2004 that targeted middle school teachers, and also provided support for IHE faculty and a limited number of fifth grade and high school teachers. The partnership includes nine local school systems—Bessemer City, Fairfield City, Homewood City, Hoover City, Jefferson County, Mountain Brook City, Shelby County, Trussville City, and Vestavia Hills City Schools—as well as two IHEs—The University of Alabama at Birmingham (UAB) and Birmingham-Southern College (BSC)—and MEC.

MEC brought their community engagement model to the partnership, which guided the development of the major components of GBMP:

- summer mathematics content courses with two days of academic year follow-up,
- development of a cadre of teacher-leaders,
- community mathematics nights to inform parents and the public about inquiry-based mathematics,
- sessions designed to inform administrators and help them support their teachers in changing their classroom practice,
- revised IHE courses at UAB and BSC for pre-service teachers, and
- a new track of the mathematics major at UAB and a corresponding new middle school mathematics teaching certification in the UAB School of Education.

During the NSF funding period, 812 teachers took a total of 1566 courses. Of these, 536 middle and high-school teachers were funded by the MSP award and 276 elementary teachers were funded locally (Malone Family Foundation, Community Foundation of Greater Birmingham, Hugh Kaul Foundation, Alabama Power, and several school systems).

The vision of MEC and GBMP was to have elementary, middle grades, and secondary teachers in every course, both to benefit from the different perspectives the teachers would bring and also to demonstrate how to teach learners with a wide range of abilities where all have access to the material, and where learners at every level are challenged to deepen their understanding of mathematics. This range of abilities was extended with the participation in the courses of IHE faculty from Education, Mathematics and Engineering who were supported by the NSF grant. Contributions from school districts and other local funds supported the participation of kindergarten through fourth-grade teachers each year. As a result the course participants ranged from kindergarten teachers to college professors.

## The GBMP Summer Mathematics Content Courses

In the pilot period and at the start of the partnership, MEC brought to GBMP three nine-day courses—*Patterns: Foundations for Algebraic Reasoning*, *Geometry and Proportional Reasoning*, *Probability: A Study of Chance*—and one 4-day course—*Numerical Reasoning*. During the funding period, *Numerical Reasoning* was extended to a nine-day course to include conceptual development of fractions, decimals, and percents, and three new courses were developed jointly by MEC, UAB, and BSC—*Patterns II: Further Explorations in the Foundations of Algebraic Reasoning, Extending Algebraic Reasoning I: A Deeper Understanding of Functions, Extending Algebraic Reasoning II: Data and Modeling*. All seven courses will continue to be offered regularly at UAB in the mathematical reasoning track of the mathematics major.

In response to a conference held by the National Academies, and after attending the first MEC course, *Patterns: Foundations for Algebraic Reasoning*, project leaders, the mathematics and teacher education faculty at UAB and GBMP and MEC instructors met to create a document that would define characteristics of challenging mathematics courses. The goal was to develop a document that would identify common characteristics of challenging courses that would support course development and instructional decision-making at every level of the educational system from primary classrooms through university-level courses. The resulting document, reproduced below, provides a framework for the work with teachers and administrators.

**Operational Definition of Challenging Courses and Curricula**
1. **Big Mathematical Ideas**
   - Teach for understanding. This refers to helping students achieve "an integrated and functional grasp of mathematical ideas."([NRC]) This includes developing conceptual understanding, strategic competence, and procedural fluency.
   - Introduce a mathematical idea by posing problems that motivate it.
   - Provide a coherent collection of problems organized around a big mathematical idea.
   - Provide opportunities for students to use multiple representations of a mathematical idea.

- Provide opportunities for students to explore real-world problems connected to big mathematical ideas.

2. **Inquiry and Reflection**
   - Engage students in inquiry.
   - Communicate that learning mathematics should be a sense-making process.
   - Ask students to justify their thinking.
   - Ask students to engage in reflection.
   - Encourage students to think critically about mathematical ideas and solutions.
   - Encourage diverse ways of thinking.
   - Communicate that both accuracy and efficiency are important.
   - Incorporate technology when appropriate.

3. **Productive Disposition**
   - Help students develop perseverance, , resourcefulness and confidence.
   - Help students become autonomous learners.
   - Provide a safe, respectful learning environment.

4. **Communication**
   - Promote the development of mathematical language.
   - Value written communication by asking students to explain their ideas in writing.
   - Value verbal communication by asking individuals and groups to articulate their thinking.
   - Value the role of communication in developing intellectual community in the classroom.
   - Establish clear expectations for mathematical assignments.

*Patterns: Foundations for Algebraic Reasoning*, is required before taking any of the other six courses. Teachers are encouraged to take multiple courses. Grade K-6 teachers are encouraged to take *Numerical Reasoning* after *Patterns* and then select from among *Geometry*, *Probability* and *Patterns II* with the *Extending Algebraic Reasoning* sequence being the culminating courses. Grade 7-20 teachers, after *Patterns*, are encouraged to take courses that match their interests and mathematical background. This paper will focus on the first course, *Patterns: Foundations for Algebraic Reasoning,* describing the course, what is unique about it, why *Patterns* was chosen as the prerequisite, and what makes this course transformational for teachers.

## Key Features of the GBMP Courses

All MEC courses offered by GBMP are unique in that they provide powerful learning experiences for elementary, middle, and high school teachers, and even university STEM faculty members. Course participants, K-20 teachers of mathematics, spend nine full days learning mathematics in a constructivist-based classroom where participants construct their own understanding of mathematical ideas while working both as independent problem solvers and in collaboration with others. In MEC courses, participants not only learn important mathematics content of relevance to students and teachers at K-20 levels, but they also learn about

mathematics teaching and learning.  Participants leave the courses knowing what it means to learn and teach mathematics in empowering ways because they have observed and reflected on their own learning, their peers' thinking, and characteristics of the environment that impacted their experiences.

Several design aspects of the course, *Patterns: Foundations for Algebraic Reasoning*, make it an optimal initial learning experience for K-20 teachers of mathematics:

## 1.  Teaching 'big' mathematical ideas
First, and foremost, the course is designed to teach teachers mathematics.  It is designed around 'big mathematical ideas' related to patterns and algebraic reasoning; mathematical ideas that deepen in complexity over time.  (The specific mathematics content of the course can be found at www.mec-math.org.)  The K-20 nature of the course allows teachers to see the trajectory of how mathematical ideas can be developed across the grades.  Professional relationships are built across the K-20 spectrum of participants.  Higher education faculty members are often fascinated, and their own work is enhanced, by seeing the visual approaches that elementary teachers frequently bring to the work.  They learn about life in K-12 classrooms as they work alongside classroom teachers.  K-12 teachers enjoy working on math problems alongside university-level mathematicians.  Collegial relationships are formed and many bridges built as teachers at every level of the system come to appreciate the work of their colleagues in producing mathematically successful students.

## 2.  Mathematics as inquiry
*Inquiry and reflection*. The *Patterns* course design is based on the belief that coming to know and understand important mathematical ideas takes time and that learning occurs through a process of inquiry where students work with each other and the teacher, as well as independently, to solve problems and to make sense of the mathematics they are learning.  As participants watch their own and their colleagues' learning develop over the course of the nine days, they come to recognize that their own students need to be given adequate time for meaningful learning as well.  They come to understand the counterproductive nature of teaching fairly isolated skills on an accelerated timeline, and they learn that going deep into the mathematical study of a few big ideas can result not only in deeper but also in broader mathematical understandings.  In short, we fully appreciate the seemingly contradictory notion that by teaching fewer mathematics topics, but teaching them more thoroughly, learners will come to understand more mathematics and to understand it as a fabric of connected and related ideas.

*Cognitive dissonance.*  The principles that guide our decision-making in MEC/GBMP courses are as follows.  First, we view confusion—the cognitive dissonance that accompanies "not knowing"—as a natural and even desirable part of the process of constructing new understandings. We work hard to ensure that course participants have opportunities to struggle with problems, to find their own ways of solving them, and to recognize that there is usually not just one way to solve a mathematics problem.  In their struggle to make sense of the problems we pose, many teachers encounter old math fears or phobias in these courses, and often even those who are not fearful still struggle with being in a state of "not knowing" as learners. We have purposefully designed courses to provide opportunities for participants to confront and get beyond their initial discomfort.  The dilemma for instructors is that we were taught that a

teacher's job is to help or teach by giving clear explanations of how to best solve problems. We have learned, however, that this natural inclination to want to put confusion to rest, and to "help" those who are struggling, is often counterproductive when it comes to developing mathematical understandings and productive dispositions.

We want to clarify our use of the word "confusion". We know that the word means different things to different people, and we don't want to leave the impression that we view all confusion as desirable. Some kinds of confusion need to be cleared up (e.g., when some kind of "social knowledge" such as how a symbol is used or how a problem is posed has not been made clear). When participants ask for help, instructors interact with them in ways that do not direct their thinking. We have come to believe that teaching by telling rarely leads to the deep mathematical understandings or productive mathematical dispositions we hope to promote. Rather than helping solve a problem for a group or individual, instructors ask probing questions to help teachers to find their own ways through the problems and honor their struggles.

*Meeting a range of learner needs*. A second idea that permeates our work is the belief that all learners are capable of mathematical thinking and of having powerful mathematical ideas. It is important for the instructor to meet a wide range of learner needs while challenging every learner. As teachers who might consider themselves "math phobic" learn to work on mathematics alongside teachers who are very confident with mathematics, including STEM faculty members at the college level, all participants learn what it is like to learn mathematics in a classroom designed to meet a variety of learner needs without labeling students. The courses are designed around expandable mathematical tasks that allow access to struggling learners yet challenge even the most mathematically sophisticated participant. Each task includes one or more challenges that encourage participants to probe deeper into the mathematics of the original problem or a related problem.

*Skills practiced within engaging and relevant mathematical contexts*. Third, GBMP recognizes that practice with new skills and concepts is essential if students are to learn how to put mathematics to work in empowering ways. In MEC/GBMP courses, such practice is always provided within engaging and mathematically important contexts that also serve to build more productive mathematical dispositions.

## 3. Mathematics as communication
Talking mathematics is the norm in all MEC/GBMP courses. From the very beginning, participants are encouraged to see and solve problems in their own ways. Processing of mathematics tasks occurs as participants work together and independently to make sense of the problems and whole group processing of experiences occurs on a regular basis. During processing, students are asked to share their diverse ways of seeing and solving problems, and reflect on ideas shared. Participants learn to consider and value diverse ways of seeing mathematics. Whole group processing is done with an eye toward clarifying the mathematics involved, finding mathematical relationships within and among the diverse ways of seeing, and learning to consider, value, and build upon each others' mathematical ideas. Participants learn to make and consider mathematically convincing arguments in ways that lay the groundwork for more formal proofs.

Participants work on tasks for as long as it takes. Everyone knows that they are not finished with a task when, for example, they have an equation for the $n$th term. Rather, they are finished when they can show how their equation describes the pattern or situation of the problem, and when they have reflected on their findings. While finding an algebraic expression may be quite challenging for some primary teachers, finding a geometric representation can be quite challenging for secondary teachers who come to the course already comfortable with numerical, tabular, or symbolic approaches to solving a problem.

## 4. Productive disposition
*Learning mathematics involves hard work.* The *Patterns* course is intellectually challenging at all times, and can be both intellectually and emotionally exhausting as many teachers struggle to learn mathematics while also striving to overcome deeply ingrained math phobias. Other teachers who are confident in their mathematical content knowledge often encounter disequilibrium when they are asked to see problems in multiple ways or to solve a problem where the solution path is not immediately obvious to them. All participants, no matter their level of competence or confidence with mathematics, are engaged with mathematical tasks that demand perseverance. Participants learn what it means to struggle and to experience the exhilaration of finally solving a problem or understanding a mathematical idea. They come to know and value that the degree of exhilaration or joy they experience in solving a problem is often directly proportional to the amount of struggle and effort expended. There are many private and public celebrations of "AHA!" moments during the course.

*Building productive learning communities.* The *Patterns* course is designed to model what it means to become part of a productive and supportive learning community. Participants come to care about each other's learning. They learn that in trying to encourage and understand the learning of others, they understand mathematics at a deeper level themselves. MEC instructors make ongoing decisions throughout the course with the goal of developing autonomous learners. Participants learn how to ask for help when they need it, and they learn how to help their colleagues, and subsequently their students, without "rescuing" them by doing the mathematical thinking for them. Teachers often come to realize that they have been rescuing their own students rather than interacting in ways that build the more powerful mathematical understandings and dispositions that would diminish the need for future rescue.

# Deepening Teacher Content Knowledge through "*Patterns*"

Many mathematicians describe mathematics as the science of patterns. A disposition to search for patterns and the knowledge to recognize and apply patterns of significance to the solving of new problems is a major goal GBMP has for teacher participants. Consequently, we chose *Patterns* as the first experience for teachers in GBMP.

The mathematical goals of the *Patterns* course are for participants to develop or deepen their understanding of the following:

- generalizations of patterns that have proven their mathematical power and significance;
- linear, quadratic and exponential functions;

- relationships among the various representations of a function–graphical, numerical, geometric, algebraic and verbal / situational;
- simplifying equations and expressions;
- multiple ways to solve a mathematical problem;
- becoming successful problem solvers who are able to use mathematics for making sense of situations and information in the world; and
- what it means to have a productive mathematical disposition.

The *Patterns* course provides an especially fertile ground for exploring the connections among graphical, algebraic, geometric, numerical, and verbal representations. These multiple representations help participants make sense of algebraic expressions. They provide verification which leads to confidence in mathematical thinking. Many participants come to an algebraic expression on their own for the first time, rather than being told the expression. For others, it is the first time they have understood many equations they "learned" in their high school and college mathematics courses.

# A Typical Day in Patterns

In an attempt to create for the reader a sense of what it is like to be a participant in the *Patterns* course, we illustrate below one day in the nine-day course.

As the 35 to 40 participants enter the classroom on about the 4th day of the course, they choose a card at random from a deck and go to the table labeled with the card value. The use of cards guarantees that participants sometimes work in random heterogeneous groups of four. Our goal in randomly grouping participants in this manner is that they learn to work collaboratively and productively with colleagues who bring diverse perspectives to each small group setting.

On the board is the outline of the day's events:

TENTATIVE AGENDA

| | |
|---|---|
| 8:30 | Number Talk |
| 8:50 | Work on Menu 1 |
| 10:45 | Processing 'Cowpens' |
| | |
| 11:00 | Break |
| | |
| 11:15 | Processing 'Robbie the Robot' |
| 11:45 | Lunch |
| 12:30 | Article Discussion |
| 1:15 | Continue work on Menu 1 (with graphing pullout) |
| 2:20 | Process 'Increasing Patterns 5' |
| 2:50 | Reflective Writing |
| | |
| HW: | Finish required tasks from Menu 1, including graphing 'Polygon Perimeters' |
| | |
| | Read assigned article. |

Below we provide a menu problem example (cowpens).  Further examples of the day's activities as outlined above are described in the appendix. A menu is a coherent collection of problems organized around a big mathematical idea. It is used to surround students with a mathematical concept that they encounter in a variety of contexts. Menus consist of both core tasks and optional ones ("desserts").  Menu tasks are posted around the room, with potentially helpful manipulatives and other resources such as graph paper and a mathematical dictionary made available.  Students choose which task to work on, spend as much time as they need to work productively on the task, and decide whether to work independently or with a classmate.

Menus are also designed to meet a range of learner needs.  The combination of core and dessert tasks is designed to ensure that all participants have challenging work to do throughout the menu time. Typically, all core menu items are posted at one time.  However sometimes a particularly challenging core task is held back and posted a few hours later so that a student does not confront it first.  Most participants might be working on tasks like the 'Handshake Problem' (how many handshakes are needed for $n$ people if everyone shakes hands with everyone else?), the 'Ice Cream Problem' (how many different double scoops ice cream cones can you create with $n$ flavors of ice cream?), or the 'Diagonals on a Polygon' problem (how many diagonals are there on an $n$-sided polygon?).  Other participants who complete the core tasks relatively quickly will be working to solve dessert problems such as Cube Pattern #4 (described below).  Because the same idea is often presented in several menu tasks in different contexts, they can be worked in any order.  Participants are not expected to finish all tasks, but are expected to complete the ones that will be processed as a group, which are announced in advance.

**A Menu Problem Example**
The two-week *Patterns* course has two menus; the first mostly consists of linear patterns with a few quadratic problems while the second menu is made up mostly of quadratic patterns with a few higher-order polynomials and exponential functions. Although many participants choose to work alone on menu tasks, the menu often starts with a group task to encourage collaboration. On this day, participants begin with a task that they are asked to work on in small groups.  The following "Cowpens" problem is posed to the groups:

The High Mountain Fencing Company is in the business of building cow pens. The company ships cow pens to all parts of the United States. Their design for a pen for one, two and three cows is shown below. How many sections of fence would it take to hold 100 cows? 1000? Any number?  The, admittedly, contrived limitation of always building long, narrow pens is imposed on this problem.

Participants are asked to give everyone a moment to think about the problem on their own. *We now ask the reader to pause and think of at least two ways in which you could solve this problem.*

After individual teachers have had time to develop their own thinking, the groups of four investigate the problem together. When a group finishes the problem, the members move on to other menu items. At a later time, the instructor will bring the whole class together to process Cowpens, and will invite volunteers to share their thinking. Most groups take about 20 minutes to solve the problem and then have about an hour and a half to work independently or with others on other menu tasks.

Whole group processing of "Cowpens"
"Processing" constitutes whole-group discussion of designated menu tasks. Participants are asked if they are willing to share their solutions. As with the number talk described above, the instructor often asks "Did anyone see it differently?" The instructor has in mind the important mathematical ideas that should come out in this discussion. Having students share different ways of seeing the solution may make it easier for students to see the connections among the solutions.

As participants share their work on "Cowpens," various expressions are examined to determine if they are equivalent to others, and just different ways of seeing the same thing. As we share Shandra's thinking below, it will help the reader to know that the instructor has previously introduced the notion of a "What-I-See" table, which is an expansion of a T-table, adding a column that describes how the user sees the geometry in the problem.

*Shandra:* I used a What-I-See table:

| Number of cows | What I See | Number of tiles |
|---|---|---|
| 1 | 3 tiles in the top row, 3 tiles in the bottom row, and just 2 tiles in the middle row, one at each end | $3 + 3 + 2$ |
| 2 | 4 tiles on top, 4 tiles on bottom, and the same 2 tiles in the middle row, one at each end | $4 + 4 + 2$ |
| 3 | 5 tiles on top, 5 tiles on bottom, and 2 tiles in middle | $5 + 5 + 2$ |
| … | | |
| $n$ | $n+2$ tiles on top, $n+2$ tiles on bottom, 2 tiles in middle | $n+2 + n+2 + 2$ |

For three cows, I saw five tiles on the top row and five tiles on the bottom row and then just two tiles in the middle row on the far left and far right. So, for *n* cows, I saw *n*+2 tiles on top and *n*+2 tiles on the bottom plus just the two tiles in the middle row for a total of $T = n+2 + n+2 +2$.

*n*+ 2 tiles on top row

| Cow | Cow | Cow |

← 2 tiles in middle row

*n*+ 2 tiles on bottom
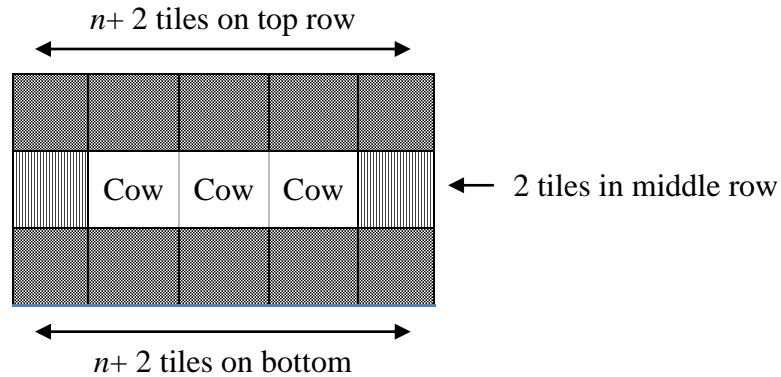
*Instructor:* What does your *n* represent?

*Shandra:*　The *n* is the number of cows and the *T* is the number of tiles needed to build a cowpen for *n* cows.

*Instructor*: Can you see what Shandra sees?  [Pause] What questions do you have for her? [Pause] Did anyone see it differently?

*Richard:*　How I saw it was similar to Shandra, but I saw 1 tile on the top and bottom of 1 cow, plus 3 on each end. Then I saw 2 tiles on the top and bottom of the 2 cows, plus the same 3 on each end. So, for *n* cows, I saw a total of $T = n + n + 3 + 3$. [Richard also shared a What-I-See table.]

*Instructor:* Does Richard's solution make sense to you?  [Pause]  Is his answer equivalent Shandra's? Can anyone convince us of that?

*Richard:*　 Sure.  I have $T = n + n + 3 + 3 = 2n + 6$ and that was the same as Shandra's answer $T = n+2 + n+2 +2 = 2n + 6$.

During processing, several varied ways of seeing emerge which provide opportunities for simplifying equations and understanding why two algebraic expressions are equivalent. Participants also share several graphs of the 'Cowpens' relationship and the instructor guides the whole group in discussing them, to make sure that certain important features of graphs are discussed.  These include lines versus points (continuous versus discrete variable), extending the graph to negative values of the independent variable (negative cows?), the meaning of where the graph crosses the vertical axis in the context of the task, the meaning of the slope, and graphing the situation versus graphing the equation.

*Instructor*: The solutions you have shared to the Cowpens problem helped us examine the connections among geometric, numerical, graphical, algebraic, and verbal representations of the problem.  [This discussion continues.]

## Participants as Learners and Teachers

In *Patterns*, teachers are focused on the mathematics nearly all of the time, but constructivist-based pedagogy is always modeled by course instructors and brought to the forefront with daily article discussions and reflective writing.  Participants are asked on the first day of class to hold their questions around practice and pedagogy until the end of the course, and they find that most of their early questions are answered through their experiences in the course.  On the final day of the course, teachers have an opportunity to ask any remaining questions they have about pedagogy and implementation of the instruction modeled in the course.  The instructor as well as participants who have routinely used inquiry-based instruction offer insights from their classroom experience.

## Assessing Learning

Participants in MEC/GBMP courses experience multiple performance-based assessments that optimize student learning including on-demand performance tasks, pre- and post-assessments, scoring rubrics, and portfolios.  Participants learn to judge the quality and thoroughness of the work they do.  Our overarching goal is to develop students who can think, reason, and use mathematics to make sense of information and situations in their world.

Two overarching beliefs guide our decision-making when it comes to assessment in MEC courses:  a good assessment task is a good learning task and multiple measures are essential.

1.  *A good assessment task is a good learning task.* If we want to know if students can use mathematics to solve new and relevant problems, then it is not adequate to simply measure their recall of skills and procedures that they have been taught directly.  Participants' abilities with skills and concepts can and should be assessed within the context of their work on meaningful performance-based assessment tasks.  Were we to measure skills in isolation, we would be promoting the idea that it is okay to teach skills procedurally and in isolation, and we believe this to be counterproductive to learning and to the development of productive mathematical dispositions.  The assessment system used in MEC courses is designed to assess to what degree a participant is able to use mathematics to solve problems.  We know that in order to put mathematics to work in productive ways, participants need to understand the mathematical relationships involved.

2.  *Multiple measures are essential*.  The common practice of measuring student outcomes by any one single test is misguided and counterproductive.  MEC uses multiple measures over time to assess participant learning.  Assessments used in the *Patterns* course are the kinds of assessments that both optimize student learning and teach participants how to learn to judge the quality of their own work in mathematics, as well as that of their peers.  In addition to continual formative assessment conducted by MEC instructors as they watch and listen to

participants work on mathematics tasks and during daily Number Talks, MEC uses multiple assessments including the following:

- Work Ethic Rubric: During the first morning of the course, after participants have been working for about an hour, they are given a rubric and asked to judge their own work ethic displayed throughout the hour. The work ethic rubric ranges from Level One: "Interferes with the work of others"; to Level Four: "Works productively and challenges self and others on assigned and self-initiated tasks; demonstrates perseverance; works collaboratively when opportunity arises; respects the rights and work of others; demonstrates craftsmanship in work when appropriate; and, takes care of materials and the environment." Participants reflect in writing on their own work ethic demonstrated during the hour. Volunteers, especially those who scored themselves less than a Level Four, are asked to share with the class what level they scored themselves, and why. The work ethic rubric is used early on in the course to establish expectations for the learning environment.

- On-demand Pre-Assessment Performance Tasks: Prior to any instruction in the Patterns course, participants are given two pre-assessment tasks. The first is a linear tile pattern where they are asked to find the number of tiles needed to build the $10^{th}$, the $25^{th}$ and the $n^{th}$ stage and explain how their answer makes sense. The second is the same task, but the pattern they are given is a quadratic pattern based on triangular numbers. Manipulatives are available and participants are given 30 minutes to solve the two tasks and explain their reasoning. Participants work independently on the task without talking with others.

- Quick and Easy Score: On day four of the course, participants select one task, do a quick score of their own work, and leave all of this work on their menus with the scored task for the instructor to collect and score over the weekend. This quick score is used by the instructor to gather information about how all participants are doing with the mathematics. It also models a way to give student feedback when teachers have lots of work to score yet want to get feedback to students overnight or on a very short timeline. Menus are returned to participants on day five of the course, and participants are given time to review instructor comments.

- Scorable Task: On day five of the course, participants are given a rubric that is designed to assess the quality of work on a mathematics task based on four traits that include understanding the mathematics of the task, the processes and strategies used to solve the task, mathematics as communication, and justification of findings. The rubric is an adaptation of the Oregon Scoring Guide that facilitates judgments on a six-level scale. It is used in MEC courses to help participants learn to judge the quality of their own work, and that of their peers. Small group and whole class discussion ensues around the rubric as participants score work on the task and debate their group scores while learning to understand the characteristics listed on the six levels of the rubric. On day seven of the course, participants select one task of their choice and write the task up as a 'scorable task' representing thorough and complete work. Each participant then exchanges papers with a colleague who has selected the same task and uses the 'scorable task' rubric to do a peer score of their colleagues

work on the task. The scored tasks are returned to their authors and pairs discuss the scoring, after which each participant does a self-score of his/her own work. Participants who also want feedback from the instructor can turn in their tasks for a third score.

- On-demand post-assessment tasks: Participants are once again given two performance tasks, one linear and one quadratic. Manipulatives are available for those who want them, and participants work independently on the tasks for approximately 30 minutes.

- Selecting Most Important Work: On the afternoon of day eight, participants are given quiet time to look over all of their work from the course, and each selects what she or he thinks is the most important piece of work. Each participant writes a cover sheet to accompany the piece, telling the instructor and other readers of their portfolio why this was the most important piece.

- Portfolio Share: On day nine, the final day of the course, participants bring their course portfolios that include: the most important piece of work; the pre- and post-assessment tasks; another scored task from the course; written reflections on himself or herself as a learner and as a teacher of mathematics; an on-demand assessment task; and a letter to an administrator or colleague synthesizing the 'big ideas' from the course and outlining the kinds of support needed as the teacher returns to the classroom and works to advance his or her practice. During the portfolio share, participants sit in groups of three of their own choosing. Each participant is given 15 minutes to share his/her portfolio. Colleagues are then given five minutes to provide feedback on each portfolio. The use of portfolios to assess learning helps both participants and the instructor assess growth in understanding of mathematics and mathematical dispositions over time. The reflective writing tasks included in the portfolios help teachers reflect on their own and their colleagues' learning and on their instructional goals for when they return to their own classrooms.

## MEC/GBMP Course Instructors

Over the course of the five-year duration of GBMP, several local teacher-leaders and higher education faculty members from the University of Alabama at Birmingham and Birmingham-Southern College have interned with MEC and now teach the courses. There are several characteristics that MEC considers essential prior to inviting individuals to intern as course instructors.

- Comfort with the mathematics of the course as demonstrated by individuals' work as learners in the course;
- Experience in teaching students within a constructivist-based classroom in ways aligned with the instructional and assessment practices modeled in MEC courses. Course instructors must be firmly rooted in constructivist and social constructivist theories about teaching and learning;

- The ability to effectively challenge even those students who come to the course knowing more mathematics than the instructor;
- A willingness to try new ideas and practices and to talk openly with participants and other MEC instructors about what is working and what is not working in their own practice;
- Demonstrated poise in working with adult learners; and
- A productive disposition as a learner who engages and stretches mathematically and who demonstrates perseverance and a love of mathematics as a participant in MEC courses.

With rare exceptions, MEC instructors first attend MEC courses as learners themselves, then are invited to attend again at MEC expense to view the course through an instructor lens. Finally, invitees attend the course a third time as interns where they share responsibility for instruction and assessment with experienced MEC instructors. Finally, if there is a good match, interns are invited to become MEC instructors. New instructors are most often asked to teach a course in a community where other instructors are teaching the same or a different MEC course so that they have colleagues they can bounce ideas off of as they teach solo for the first time.


## Evidence of Impact

The research team used four measures to assess the impact of *Patterns* on teachers' content knowledge: a pre-post performance assessment, a modified version of the Learning Mathematics for Teaching Project's *Test of Content Knowledge for Teaching Mathematics-Patterns* (CKTM-*Patterns*), a rubric-scored course portfolio, and an observational behavioral checklist.

Teachers were scored on a pre-post performance assessment task using an adapted version of the six-point Oregon Department of Education scoring rubric for performance assessments in mathematics. The median score increased two points on most rubric dimensions. A Wilcoxon Signed Ranks Test showed the increases from pre-assessment to post-assessment on each dimension to be statistically significant ($p \leq 0.05$).

In addition to the performance assessment, the CKTM-Patterns was administered to teachers at the beginning and end of the two-week *Patterns* course ($N=314$). There was a 3-point mean score increase (out of 31 items) from pre to post administration, which yielded a 0.5 effect size. The CKTM-*Patterns* was administered longitudinally to a sample of teachers ($N=35$) following a second or third course (whose content was unrelated to *Patterns*). There was a pre-post mean increase of 2.86 points, and a pre-post longitudinal mean increase of 3.58 points. This evidence suggests that content acquired in the *Patterns* course is sustained, even improved, over time.

Other evidence of teacher understanding can be derived from analysis of portfolios generated during summer courses, which contain reflective pieces by the teacher, self-selected pieces of work, pre- and post- assessments, and a summative on-demand task. A sample of portfolios was rated each year on a four-point rubric with five dimensions—problem translation, mathematical procedures, productive disposition, inquiry and reflection, and justification and communication. Frequency results of consensus judgments among three raters and median scores on each

dimension indicated more than half of the sample demonstrated performance that was at or above level 3 on each dimension.

An observational behavioral checklist was used to monitor changes in teacher participants' behavior throughout the two weeks.  In years 3-5, three course participants were chosen at random from each section of *Patterns*.  Each was observed three times over the two-week course—on the first day, on the fourth day, and on the eighth day.  Observations took place when participants were working in groups or working on tasks with other participants. The table shows the number and percentage of participants who exhibited a given behavior at a given time for *N*=30 teachers.

It appeared that meaningful, observable change in teachers' mathematics understanding and communication occurred in that second week of instruction, particularly in the areas of productive disposition and communication. Most behaviors were observed for fewer than 5 of the participants on the first day. For most of the behaviors on the checklist, fewer than half of participants demonstrated evidence by the end of the first week of instruction.  However, by the end of the second week, there was evidence of all behaviors from a majority of participants.

| CCC Dimension | Day 1 | Day 4 | Day 8 |
|---|---|---|---|
| **Understanding of Mathematical Ideas** | | | |
| Uses variables to describe unknowns | 4 (13%) | 12 (40%) | 29 (97%) |
| Explains why equations make sense geometrically | 2 (7%) | 9 (30%) | 23 (77%) |
| Represents linear and quadratic equations in variety of ways | 0 | 5 (17%) | 19 (63%) |
| **Productive Disposition** | | | |
| Persists when answer is not known | 0 | 13 (43%) | 28 (93%) |
| Asks for guidance but not answers | 3 (10%) | 9 (30%) | 26 (87%) |
| Tries variety of strategies for approaching problem | 3 (10%) | 20 (67%) | 28 (93%) |
| **Inquiry and Reflection** | | | |
| Makes extensions and connections beyond immediate problem | 0 | 5 (17%) | 22 (73%) |
| Explores why it works and whether it will always work | 0 | 2 (7%) | 16 (53%) |
| Confusion and mistakes lead to further exploration | 6 (20%) | 20 (67%) | 30 (100%) |
| **Communication** | | | |
| Explains reasoning fluently | 0 | 5 (17%) | 24 (80%) |
| Asks probing questions | 6 (20%) | 11 (37%) | 28 (93%) |
| Shares ideas with class | 12 (40%) | 17 (57%) | 29 (97%) |

# Lessons Learned

Our partnership learned valuable lessons which allowed us to improve our content courses. First, we found that graphing was a weakness for many participants, even at the end of the *Patterns* course. *Patterns* has two Menus and, initially, graphing was only incorporated into Menu 2. To address the issue, we intentionally asked participants to graph patterns from Menu 1 and, during whole-class processing, emphasized the connections among the graphical, algebraic, numerical, geometric, and verbal descriptions of the patterns.

A second finding was that, when asked to build geometric patterns and search for a relationship between the step number and the total number of tiles required to build that step of a pattern, participants are often drawn to a recursive formula that describes the number of tiles required to build the *nth* step, $T(n)$, in terms of the number of tiles required for the previous step, $T(n-1)$. Sometimes participants focused on a recursive formula to the extent that it inhibited their success in finding a closed-form formula that describes $T(n)$ in terms of the step number *n*. To address this, we changed the format of the problems. Originally, we had listed the first three or four steps of the pattern in order and asked the participants to build the next two steps and, eventually, to find the number of tiles required to build the $n^{th}$ step. We redesigned the tasks so that we provide several steps that are neither consecutive nor in numerical order (for example, see the "Increasing Patterns 5" in appendix). We found that breaking the mold of building the pattern in order, coupled with the introduction of expanded 'What I See' tables described above emphasized the relationship between the step number (the input) and the number of tiles (the output) and helped participants to find a closed-form formula for themselves.

At the onset of the project, GBMP intended to develop three new courses, the first of which was to be *Extending Algebraic Reasoning*. This course was intended to extend and deepen an understanding of algebra and continue to emphasize the connections among different representations while making use of graphing calculator technology. The first summer the new course was offered, we found that, for many participants, there was too big a jump in the level of material between the *Patterns* course and *Extending Algebraic Reasoning*. To address this concern, we developed an intermediate course, *Patterns II*, to provide more experience with the material in the *Patterns* course and extend the material to include transformations of linear and quadratic functions and more experience with graphing and symbolic manipulation. The *Patterns II* course has turned out to be a great success and very popular.

A final change in terms of course offering concerns the Probability and Data Analysis course. Initially, MEC had one nine-day course designed to cover both topics. However, we learned that teachers needed more time to develop a deep understanding of the probability topics and so we revised the course in order to focus exclusively on probability for the full nine days. MEC's 5-day Data Analysis course will be offered for the first time in summer of 2011.

The final lesson learned focuses on pedagogical knowledge. Two important and intertwined goals are for participants to deepen their content knowledge and experience the power of an inquiry-based learning environment. Both objective and performance-based tests of content knowledge provided evidence of growth in content knowledge. Qualitative measures indicated that most teachers found experiencing a challenging, inquiry-based learning environment first-hand to be a transformational experience and reported the desire to radically change their own

classrooms to provide this type of experience for their students. However, after they returned to their schools, teachers faced barriers to implementation.

Although most teachers left the *Patterns* course with the desire to provide an inquiry-based learning environment for their students, a major internal barrier is that they did not feel confident about their ability to provide an inquiry-based approach to their course of study. Not having seen a model of a school year taught in this way and not having tried to lead mathematical discussions through questioning left many teachers feeling that they needed more practical information in order to implement inquiry-based instruction. To address this barrier, in the final year of GBMP Phase I, twelve teacher-leaders began facilitating school-based professional learning communities (PLCs) focused on implementing inquiry-based instruction. These PLCs studied articles, books and video clips to facilitate professional conversations focused on improving mathematics instruction in their schools. In addition, they recognize the need for assessment practices to become more aligned with inquiry-based instruction. The combination of an intensive two-week mathematics course during the summer and school-based professional learning communities during the academic year provides teachers with the opportunity to deepen their own content knowledge and support for providing a powerful learning environment for their students.

In addition to the internal barriers, participants identified specific external barriers to implementation. These included a lack of curricular materials aligned with inquiry-based pedagogy, administrators who do not actively support inquiry, concerns that parents might react negatively to change, and the pressure to cover material associated with high stakes testing. GBMP recognized that these are barriers outside of the teachers' control. MEC's community-engagement model attempts to address some of these factors by regularly holding sessions for parents and administrators. Parents overwhelmingly rate these sessions as informative or extremely informative. These sessions provide parents a glimpse of what it means to understand and make sense of mathematics. The true power of what it is like to learn mathematics in an inquiry-based learning environment, however, is described in numerous teacher portfolios with comments such as:

- *"This course was such an enlightening and beneficial experience. I look at math in a new light. It makes so much more sense now."*

- *"The teacher provided a <u>life-changing</u> curriculum framework as she modeled applications, questioning, management, and analyses in an engaging environment."*

## Appendix

In this appendix, we continue our description of a typical day's activities as outlined on page 9. The typical day begins with a Number Talk.

**Multiplication Number Talk**
The day begins with a Number Talk on multiplication (previous day's Number Talks have addressed addition and subtraction). Increasing participants' numerical reasoning is a priority goal in the *Patterns* course, and Number Talks are a major vehicle for accomplishing this goal. During Number Talks participants work to solve computational problems mentally. They are encouraged to solve problems in their own ways. When everyone has had time to think about a given problem, they share their methods and solutions with each other, and examine mathematical ideas as they reflect on and talk about their varied solutions. Flexible thinking, efficiency, and accuracy of results are emphasized, and mathematical connections between different solutions are investigated. In addition to learning to compute flexibly, participants build foundational skills and understandings of mathematical concepts such as place value and the arithmetic properties as they 'talk mathematics' on a daily basis. Flexible numerical reasoning is the goal. For example, the instructor poses the following problem on the overhead:

$$16 \times 27$$

*We ask the reader to pause to think of at least two ways of solving this problem mentally before reading on.*

Computation problems are often, but not always, posed horizontally so as to *not* predispose the participants to thinking in terms of standard algorithms that arrange the numbers vertically. After giving sufficient time for all participants to solve the problem without writing anything down on paper, the instructor asks for answers, writing each on the board (including incorrect answers, if any). Then the instructor asks if someone will share their strategy for solving the problem (and if there are multiple answers, asks which answer they are defending). Tenisha defended 432.

*Tenisha*:   I changed $27 \times 16$ to $(25 + 2) \times 16$.
          $25 \times 16$ is a quarter of 1600 or 400.
          $2 \times 16$ is 32.
          So $400 + 32$ is 432.

*Instructor*: Does anyone have a question for Tenisha? [Pause] So you found it easier to think about $27 \times 16$ instead of $16 \times 27$? Will that always work? Did anyone notice what property Tenisha was using here? [Several participants reply that it is the commutative property of multiplication.]

*Instructor*: How about when Tenisha thought about 27 as $25 + 2$ and then multiplied each part by 16? [The instructor writes the equation:]
        $(25 + 2) \times 16 \ = \ (25 \times 16) \ + \ (25 \times 2)$
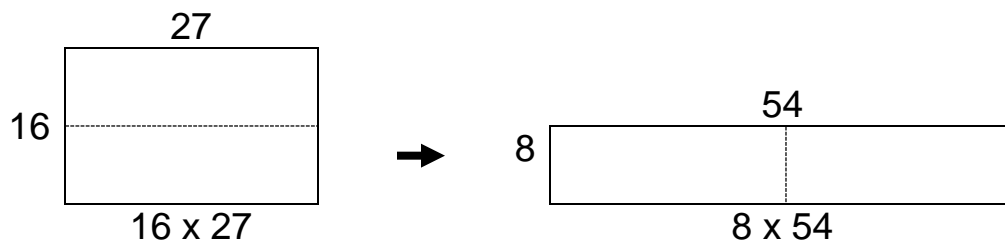
        Will that always work? How do you know?

*Dennis:*    Tenisha was using the distributive property when she broke the 27 apart [Dennis drew the diagram below.]

| | 25 | 2 |
|---|---|---|
| **16** | 400 | 32 |
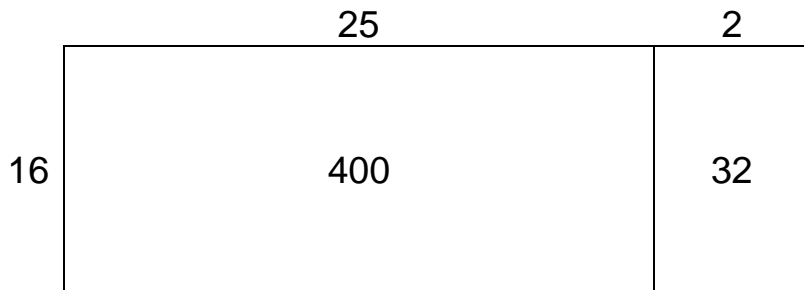
*Instructor:* Number Talks provide a strong foundation for success with algebra.  When children are used to breaking numbers apart and putting them back together, they are using the same properties that are used in algebra. When they get to algebra, children continue to put arithmetic properties to work with variables as they have been doing with numbers for a long time during Number Talks.

*Instructor*: Who solved this problem in a different way? [Pause]

*Tonya*:    I halved 16 and doubled 27 to change the problem to 8 × 54.
Then I did 8 × 50 is 400 and 8 × 4 is 32.
I added 400 and 32 to get 432.
[Tonya drew the following diagram to explain her method for solving the problem.]

27 / 16 / 16 x 27 → 8 / 54 / 8 x 54

*Instructor*: Does Tonya's solution make sense? [Pause] Will her method always work? [Pause] How do you know? [The reader may have observed that multiplicative inverses and multiplicative identity are at play here.  This will be discussed in future Number Talks.]

Number Talks are a daily fixture in the *Patterns* course.  The open array models above lay the foundation for later Number Talks around algebraic expressions such as $(x + 1)(x + 2)$.

| | $x$ | 1 | 1 |
|---|---|---|---|
| $x$ | $x^2$ | $x$ | $x$ |
| 1 | $x$ | 1 | 1 |
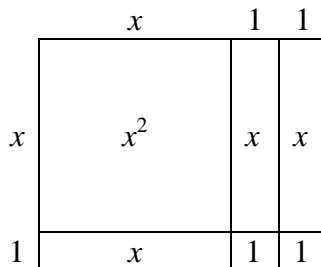
**Processing 'Robbie the Robot'**
The Robbie the Robot appears on Menu 1, and participants were told the previous day that the problem would be processed today.

"Below are 1 year old, 2 year old, and 3 year old pictures of Robbie. Build Robbie at age 4. Describe in words what Robbie would look like at age 10. What does Robbie look like at age $n$ and how many blocks does it take to build him?"
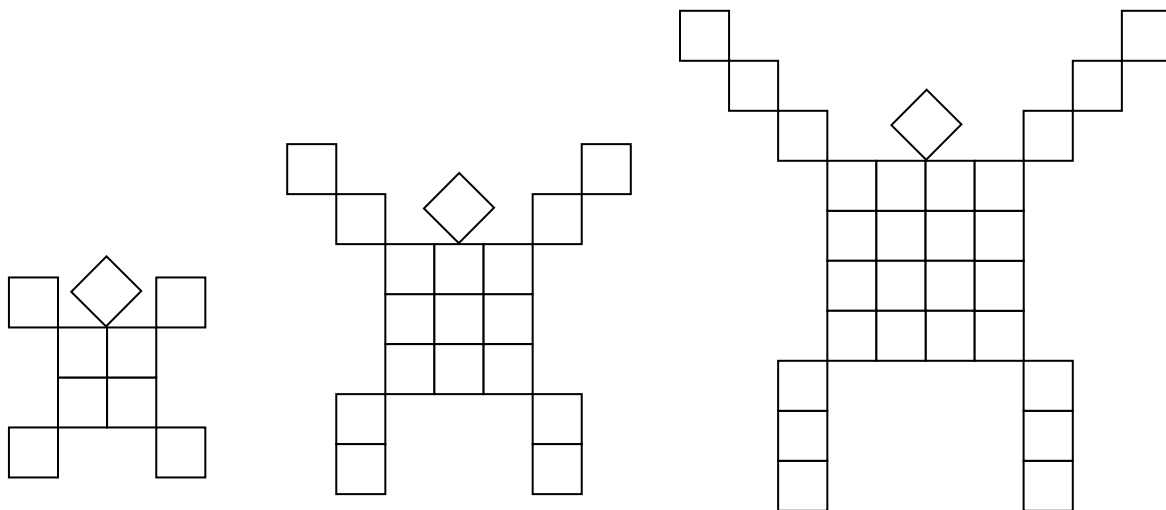


Students will be told during menu time that they should also graph Robbie's three parts—head, limbs, and body—on the same axes. In processing Robbie, equations and graphs for each of his parts will come up, contrasting constant (the head), linear (the limbs), and quadratic (the body) functions. If it does not come up on its own, the instructor will ask the class to predict what the graph of the total number of blocks needed to construct Robbie would look like.

**Lunch**
It is not unusual during the lunch hour for teachers to stay in the classroom to eat so that they can work on a menu task they can't let go of. By the time the instructor returns from lunch, nearly everyone is back working on menu tasks without having been directed to go to work.

**Article (homework) Discussion**
After lunch, the whole class discusses an article that was assigned for homework the previous evening, entitled, "Constructivism: A Theoretical Revolution for Algebra," by Donald Blais (1988). Homework reading assignments are purposefully structured to ensure that participants dig deeply into the readings. Teachers read assigned articles daily and come to class prepared to share in small groups what they determine to be three most important ideas in the assigned reading. Small groups then work together to come to consensus about the three most important ideas they as a group will contribute to the class discussion that ensues. Articles are chosen that help teachers better understand the theoretical underpinnings, challenges and benefits of constructivist-based mathematics classrooms. Reading these articles often prompts teachers to reflect on their own classroom practice and class discussion involves both theoretical elements and practical concerns that teachers have about implementation of constructivist techniques in their instruction.

**Continue work on Menu 1 (with graphing pullout)**
Since participants work alone or with a colleague during menu time, the instructor is often available to work with individual students or small groups. During this menu time, the instructor offers a small group pull-out session on graphing for those who want more help with that topic. Participants who are interested in the announced topic self-select to join a pull-out group. Today's pull-out session focuses on graphing some of the functions that arise in the menu items. Another day, the pull-out session is designed for teachers who want additional support in finding algebraic representations of linear patterns. Pull-out sessions are one way the instructor differentiates instruction by providing extra help tailored to the needs of various individuals. The dessert tasks provide another means of differentiation during menu time for those who seek additional challenges.

**Processing 'Increasing Pattern 5'**
A few stages of 'Increasing Pattern 5' are shown below and participants are asked how many tiles it would take to build any building (adapted from: *Developing Number Concepts Using Unifix Cubes b*y Kathy Richardson). The reason the buildings are not listed as stages (step number) 1, 2, 3, etc., in order, is to try to *not* habituate the student to think only of the recursive relationship between successive stages.



|      Stage 7      |      Stage 4      |      Stage 6      |
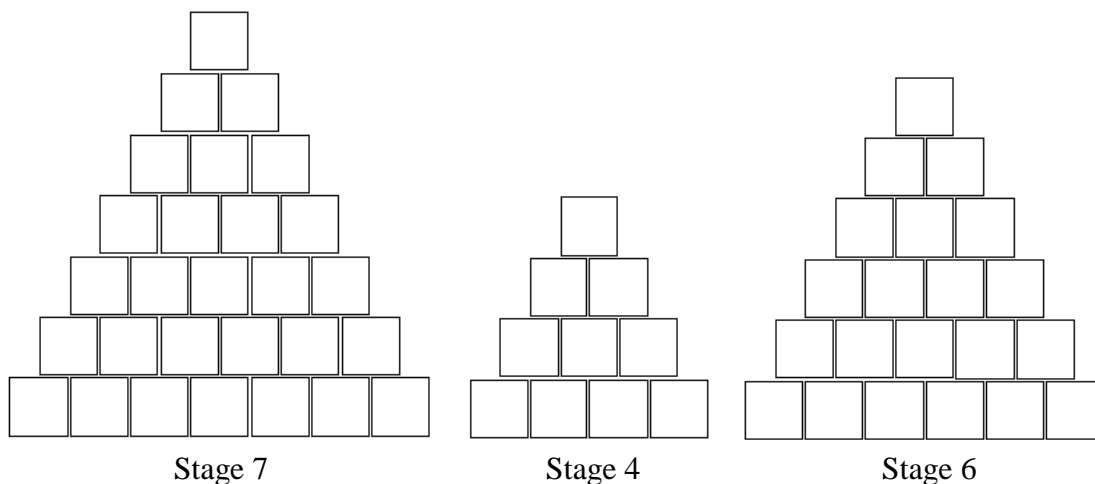
Some participants solved this problem relatively quickly while others worked on it for several hours over 2- 3 days. The flexibility of the menu format allows participants to spend varied amounts on this and other tasks. Whole group processing happens when everyone has time to solve the problem. As usual with whole class processing, the instructor starts by asking for volunteers to share their work.

*Patricia:*   I was building stage 3, moving tiles around, and I realized I could "left justify" stage 3 to look like this (the diagram on the right, below).
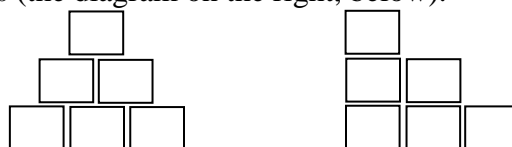
Then I put two copies of stage 3 together like this [see below]. Now it's easy to count that there are $3 \times 4$ tiles in all, but that's twice as many as I wanted, so there's really only $(3 \times 4)/2$ tiles in stage three. For stage n there would be $[n \times (n+1)]/2$ tiles.
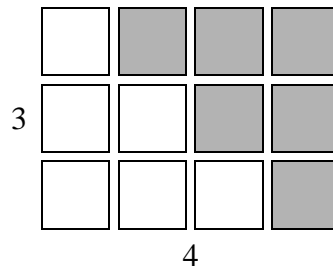


Other participants used Gauss' theorem for adding consecutive integers, and still others solved the problem by completing a square with color tiles, dividing the square in half, and adding back ½ of each tile on the diagonal for a result of

$$\tfrac{1}{2} n^2 + n/2$$
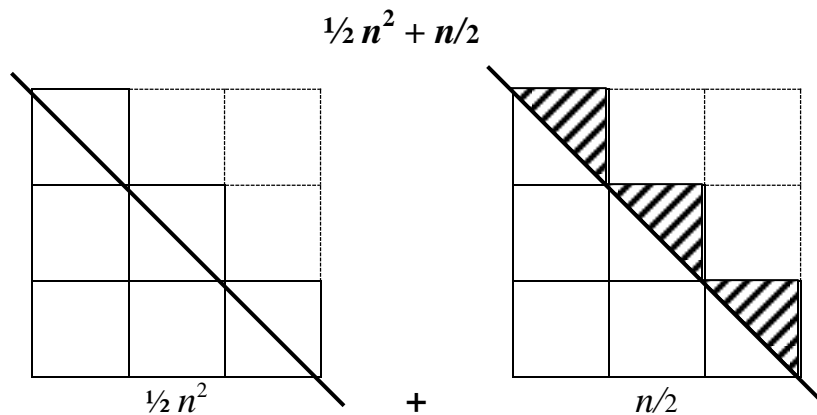


A participant asked about similarities he observed between two different solutions and this discussion continued.

The pattern above is an example of an expandable task. An initial access point is to build several additional steps in the pattern, further exploration allows one to determine the number of tiles needed for the 10th and 100th steps, and a more algebraically sophisticated problem is to find the number of tiles needed for the $n$th step. This problem is further expanded upon in a dessert task that is a three-dimensional analog of the pattern (called Cubes Pattern #4). Several participants including high school and university mathematics faculty members wrestled with this problem for hours. Since not all participants work on dessert tasks, they are not processed with the whole class, but discussion within small groups or with the instructor often occurs during menu time.

**Reflective Writing**
At the end of the day, the instructor gives participants a reflective writing prompt:

Spend a few minutes writing reflectively about your experiences in this course so far.
Your reflections might include:
- new ideas you have encountered
- shifts in your thinking

        -successes you'd like to share
        -areas of dissonance
        -things you are continuing to ponder
        -anything else you'd like to share…

The purpose of this assignment is to give participants a chance to reflect on their learning experience during the day.  For some participants, inquiry-based learning is a huge shift from their previous experiences with learning mathematics.  Reflective writing gives them a chance to think about the specifics of their learning, e.g., what it feels like to work in a small group; what do you do when you hit a road block and are not sure what to do next in a problem (what actions on the part of the instructor or other participants or yourself helped you get beyond the roadblock); when someone needs help, what are effective ways of providing it?  After participants have had a few quiet minutes to write, the instructor asks if there is anyone who would like to share their reflections with the whole class.

Although this is where we end our visit to the *Patterns* classroom, if you returned a few days later, you would find participants working on the 'Handshake Problem', the 'Ice Cream Problem', or the 'Diagonals on a Polygon' problem.  You might hear a teacher working on the ice cream problem exclaim "this is like the handshake problem but it starts in a different place!"  The menus are purposefully designed so that participants confront the same mathematical idea in different contexts.  For most participants, a familiar problem set in an unfamiliar context genuinely feels like a completely new problem.  By making connections between related problems and synthesizing the underlying mathematical ideas, teachers build deep understandings and bring that confidence with them as they attack new problems and continue their search for patterns, laying the foundation for algebraic reasoning.